Optimal Risk Sharing and Incentive Provision in Social Security Systems: A Mechanism Design Approach*

Rafael Berriel — *Stanford University*

Carlos E. da Costa — *EPGE FGV*

July 25, 2023

Abstract

This paper investigates how risk should be shared between workers who require incentives and retirees who do not. We analyze the impact of incentives on workers' ability to bear risk and examine how the timing of incentives through entitlements should vary depending on the state of the economy. Our findings suggest that perfect risk sharing is optimal when the utility from consumption is logarithmic or when aggregate productivity growth is independent and identically distributed. Our numerical analysis suggests that the deviation from perfect risk sharing is small. We relate the quantitative findings to the failure of a consumption-based stochastic discount factor (SDF) to price economic growth, which is reminiscent of the equity premium puzzle. Once we augment our model with shocks that generate volatile enough SDF's, numerical deviations from perfect risk sharing are substantial. ***Key-words:*** *Social Security; Inter-generational risk sharing; Heterogeneous workers.* ***JEL Codes:*** *D13; H21; H31.*

*__Berriel (rberriel@stanford.edu)__: *Stanford University*; **da Costa (carlos.eugenio@fgv.br)**: *EPGE FGV Brazilian School of Economics and Finance*; Praia de Botafogo, 190 - 1105, Rio de Janeiro/RJ - Brazil. da Costa thanks the hospitality of Banco de Portugal and CNPq project (304955/2022-1) for financial support. This study was financed, in part, by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. We thank Nicolas Caramp, Isabel Correia, Thorsten Drautzburg, João Guerreiro, Felipe Iachan, Matthew Jackson, Ravi Jagadeesan, Peter Klenow, Patrick Kehoe, Humberto Moreira, Elena Pastorino. Cézar Santos, Ilya Segal, Dejanir Silva, Pedro Teles, Eduardo Zilberman, and seminar participants in the 2019 Lubramacro and SBE Meetings, the 2020 Econometric Society World Congress, Banco de Portugal, EPGE FGV Theory Group and Stanford Economics Theory Group for their invaluable comments. All errors are our responsibility.

SOCIAL security systems generate entitlements that must be met by resources produced by future generations. The set of rules that govern these entitlements entails a variety of risk-sharing arrangements between workers and retirees. While it is often the case that adjustments to pensions and contributions are made 'ex-post' through political bargaining, this is clearly sub-optimal. As suggested by Barr and Diamond (2009), "[...] with social insurance (partial or total) tax finance, the institutional structure should be designed explicitly to spread risk, rather than being an afterthought for poor outcomes."

But how ought risk be optimally shared between workers and retirees? To answer this question we need a framework that takes into account each group's capacity to carry risks and the labor market consequences of the potentially distortionary financing of such a system. The purpose of this paper is to provide one such framework.[1] We write the simplest possible model in which workers, who must be incentivized, and retirees, whose entitlements were designed, in part, to provide incentives when they were workers, must share the risk of economic growth.

Aggregate shocks are added to an overlapping generations version of Mirrlees' (1971) economy – see Ordover and Phelps (1979); Golosov et al. (2003); Grochulski and Kocherlakota (2010); Farhi and Werning (2013). Information frictions and intra-generational distributive motives generate the incentive problems faced by the policymaker. A mechanism design approach allows us to address risk-sharing arrangements and the distortions needed to finance them without imposing any ad hoc restrictions on the institutional setting in which choices are made.

The driving forces determining optimal risk-sharing between workers and retirees have two parts. First, workers and retirees differ since workers need to be provided incentives while retirees need not. Incentive provision in turn requires consumption to vary across agents since those who produce more must be rewarded for doing so. The dispersion in consumption required for the provision of incentives affects the marginal value of resources for the incentivized group. If incentive provision is harder in some states of the world than in others, then the value of resources assigned to workers will vary, which is in contrast with retirees whom one need not incentivize.

This is not the end of the story, though. Workers will eventually retire. Second-best principles suggest that the government should optimally spread the reward required to incentivize them in two parts: current earnings, and promises of future benefits. The

[1] Of course, we claim not that we are the first to address intergenerational risk sharing – see Enders and Lapan (1982); Gordon and Varian (1988); Ball and Mankiw (2007); Barr and Diamond (2009); Krueger and Kubler (2006). We innovate by placing incentives and the interaction with government distributive motives at the forefront.

optimal backloading of incentives must, therefore, be added to the analysis.

We show that the two forces aforementioned can be nicely captured in two very simple expressions. The first expression shows that the fraction of total consumption allocated to workers depends on a moment condition which depends on this group's cross-sectional consumption dispersion.[2] This moment condition captures how hard it is to provide incentives. When it varies across states of the world, then at the optimum, risk sharing is not perfect. The second expression encodes the value of backloading of incentives, by showing its equivalence to a reduction in the dispersion of types in a static economy.[3] If the dispersion of types is reduced by backloading in a state-dependent way, the moment condition will also vary across aggregate states, thus leading to imperfect risk sharing.

Both the risk-sharing and the backloading expressions are written as a function of endogenous variables. Yet, in an endowment economy, if there is perfect risk-sharing at the optimum, the constrained efficient allocation displays a separable structure under which the replacement ratio has a very simple parametric expression. This expression can then be used to derive sufficient conditions on primitives for perfect risk sharing to be constrained-efficient. It happens if either utility is logarithmic or the exogenous growth in productivity follows an i.i.d. process.

Although away from perfect risk-sharing, the efficient replacement ratio does not land itself to an analytical expression, the backloading expression at the perfect risk-sharing allocation is still useful. We perturb an economy for which perfect risk-sharing characterizes the optimum by introducing a small degree of persistence in aggregate shocks. The adjustments made to preserve incentive compatibility, and resource feasibility at minimal welfare cost (or maximal welfare gain) provide a local result regarding the departure from perfect risk-sharing. We find that starting from the perfect risk-sharing rule it is optimal to assign more risk to retirees (resp. workers) if the productivity growth process is mean reverting (resp. persistent).[4]. Optimal incentive provision requires consumption dispersion and positive labor wedges. The harder it is to provide incentives, the greater the dispersion of consumption and the larger the labor wedges. Since the separable structure that characterizes perfect risk-sharing implies state-invariant labor

[2] To be precise, dispersion is measured by the σ^{th} moment of the distribution of consumption shares, where σ is the agents' coefficient of relative risk aversion.

[3] Backloading is expressed as a simple function of pensions replacement ratio.

[4] Around perfect risk sharing the backloading increases with expected productivity growth when $\sigma < 1$. This leads to lower consumption dispersion. However, because the moment condition decreases with dispersion when $\sigma < 1$ and increases when $\sigma > 1$, we find that it is optimal to increase the risk born by retirees when there is mean reversion and to reduce when growth is persistent, independently of σ.

wedges, we find that the introduction of persistence in the productivity growth processes makes it optimal to make wedges pro-cyclical whereas mean-reverting growth processes are associated with counter-cyclical wedges.

We run numeric exercises that lead to optimal allocations that have the same properties uncovered by our local perturbations. When we parametrize the economy with values that are similar to the ones typically used in the literature, the constrained efficient allocation exhibits (near) perfect risk-sharing. What is going on? As in Veracierto (2020), deviations from optimal risk-sharing are so small that the slackness in the incentive constraints[5] is not distinguished from zero from a numeric perspective. Because benefits do not display memory, and because these entitlements are the only possible source of memory, the whole allocation is devoid of memory, thus (approximately) displaying perfect risk-sharing.

We relate these small quantitative impacts to the failure of a consumption-based stochastic discount factor (SDF) to price economic growth. These findings are reminiscent of the equity premium puzzle. We make this explicit by considering a variation of our model with taste shocks that generate an SDF that can account for the equity premium. In this case, the deviations from perfect risk sharing are far from trivial.

The lesson seems to be that perfect risk sharing is approximately optimal for the relevant range of parameters used in the macroeconomics literature.

Finally, we present two simple mechanisms that can implement the optimal allocation: 1) a pay-as-you-go system with compulsory contributions to the Social Security System. 2) A Capitalization System where agents choose how much to save when they are young and receive retirement payments proportional to their savings. For the Capitalization System, the return on the investment (net of taxes) is contingent on the aggregate state of the economy. Interestingly, in many countries, pension funds are required to invest in safe assets. If we take safe assets to mean risk-free assets, then taxes on their return must be state-contingent, otherwise, efficiency will not be attained.

The rest of the paper is organized as follows. After a brief literature review, Section 2 describes the environment, the planner's objective, and the main definitions used in the paper. In Section 3 the main theoretic findings are displayed. We perform a numeric illustration in Section 4 and discuss implementation in Section 5. Section 6 concludes the paper. Proofs are collected in the appendix.

[5] See discussion before Proposition 5.

Closely related Literature

The ex-ante welfare perspective taken in this paper is also used for the study of inter-generational risk sharing by Gottardi and Kubler (2011). They assume that cohorts are comprised of homogeneous agents and do not consider incentive provision, focusing, instead, on the general equilibrium effects of different risk-sharing arrangements. Ball and Mankiw (2007), too, addresses the inter-generational risk-sharing problem but from a partial equilibrium perspective.

Regarding both the policy question and the environment, Enders and Lapan (1982) are closest to our setup. They consider a Diamond (1965) OLG economy: with two periods lived cohorts who work in the first period and consume in the second. Social security contributions may distort labor supply decisions, which is something we also allow for when we impose incentive constraints on workers. Enders and Lapan's economy is one with homogeneous agents where ad hoc tax instruments are used with the sole purpose of financing social security benefits.[6]

Mirrlees' and Diamond's frameworks, thus freeing the analysis from ad hoc restrictions on policy instruments, were first combined in Ordover and Phelps' (1979) analysis. They considered a deterministic setting which means that no risk-sharing discussion was possible.

To the best of our knowledge, it is Phelan (1994) who first adds aggregate shocks to a dynamic agency setting. Phelan considers an overlapping generation model with aggregate uncertainty but his model differs from ours in two relevant dimensions. First, he supposes that effort is chosen *before* the aggregate shock is revealed, while in our model, the effort is chosen *after*. This timing difference has a crucial impact on the properties of allocations in which we are interested since, in his model, the effort can only be conditioned on the expected value of the current shock, not the shock itself. Unless there is persistence in aggregate shock, the incentive provision will be, by construction, state independent. In our model, the Planner can choose whether to provide incentives that are state-dependent or not. Second, while using an overlapping generation model, the space of actions - effort - from the agents is the same in every period; Phelan's is a perpetual youth model in which agents never retire. It is not possible to discuss social security. Finally, he considers a moral hazard problem, while we focus on the intra-generational redistribution problem in a screening context.

Werning (2007) introduces aggregate shocks to a dynamic Mirrlees' model. Agents

[6] Our approach separates the tasks of deriving optimal allocations from implementation – see Section 5. In Enders and Lapan (1982), the question of welfare improvement is instrument-dependent.

are infinitely lived. Hence, there is no scope for discussing inter-generational risk-sharing. da Costa and Farinha Luz (2018) generalized Werning's (2007) permanent type assumption, considering all possible intertemporal correlations of idiosyncratic types. They have shown that, for any structure of idiosyncratic shocks, log preferences are sufficient, in an endowment economy, to generate perfect risk-sharing and allocations that do not depend on past realizations of aggregate shocks. For any other iso-elastic preference, state and history-dependent labor wedges characterize the optimum. They focus on a single generation, which precludes a discussion of intergenerational risk-sharing. In our main specification, agents have a single working period, which means that they face no idiosyncratic uncertainty, as in Werning (2007). However, every period a new generation is born, which means that incentives must be provided as if there were idiosyncratic i.i.d., shocks, as in Veracierto (2020).

Veracierto (2020) also uses a dynamic Mirrlees' model enriched with aggregate shocks and capital stock to understand how incentive provision affects the optimal distribution of consumption between types and over time. When restricted to log preferences, Veracierto proves an irrelevance result that allows one to treat the idiosyncratic incentive provision problem and the aggregate allocation problem separately. Numerical simulations indicate that the irrelevance result holds approximately for more general preferences. Veracierto's is a perpetual youth model.

Demange (2008) explicitly focuses on how moral hazard affects risk sharing between groups. She shows that the way relative (among groups) consumption dispersion varies across states of nature determines risk-sharing rules. She also emphasizes the role of the timing of effort that we have mentioned in reference to Phelan's (1994) work. Her model is static thus precluding the discussion of incentive backloading which is at the heart of our story.

Scheuer (2013) also uses a dynamic moral hazard model with aggregate shocks. There, the aggregate shock is mean-preserving and only affects the ability distribution. He shows that, when asset trades must be taxed, the marginal tax will be positive for securities that pay in states where the ability distribution is "riskier". In our paper, we show how tax movements affect and are affected by consumption dispersion, an essential channel to understanding aggregate risk-sharing.

2 The Environment

The economy is comprised of overlapping generations of finite-lived agents. Preferences are strongly separable across periods and between consumption and effort. Agents are homogeneous concerning everything but the disutility they suffer from exerting effort.

The two periods lived agents have preferences regarding deterministic vectors of effort, e, consumption at youth, c^y, and consumption at old age, c^o, of the form

$$\mathscr{U}(c^y, c^o, e) = u(c^y) - \theta h(e) + \xi\beta u(c^o),$$

where

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \quad \text{and} \quad h(e) = \frac{e^\gamma}{\gamma}, \quad \text{for } \sigma > 0, \sigma \neq 1, \gamma > 1,$$

$u(c) = \ln c$ for $\sigma = 1$.[7] And ξ is a random variable taste shock.

At birth, each agent draws θ from a distribution $F(\theta)$, with density function $f(\theta)$. $\theta \in \Theta = [\underline{\theta}, \overline{\theta}]$, defines the utility cost $\theta h(e)$ incurred by the agent who exerts effort e. The draws are independent across agents, and across cohorts. We rely on the law of large numbers so that the cross-sectional distribution of types is the same for every cohort.

For most of what follows, we consider an endowment economy such that with a We consider a general technology, $Y_t = zN_t$, where N_t is the aggregate supply of efficiency hours in period t and z is the current period productivity shock. We write z_t to denote the economy's productivity in period t, and $z^t = (z_1, ..., z_t)$ to denote the history of productivity shocks.

An ***allocation*** $(c^y(\theta, z^t), e(\theta, z^t), c^o(\theta, z^{t+1}))_{\theta\in\Theta,t,z^t}$ is a measurable mapping from aggregate history to an assignment of young age consumption, effort and old age consumption for all types of each cohort. An allocation is ***feasible*** if for every period, t, and every history, z^t,

$$\int_\Theta \left\{c^y(\theta, z^t) + c^o(\theta, z^t)\right\} f(\theta)d\theta \leq z_t \int_\Theta e(\theta, z^t)f(\theta)d\theta. \tag{1}$$

Let $(c^y(\theta, z^t), e(\theta, z^t), c^o(\theta, z^{t+1}))_{\theta\in\Theta}$ denote the allocation for cohort t if the economy experiences a history z^t. Then the utility attained by a θ-agent who belongs to this

[7] Some of our results extend to other $\mathbb{C}^2$ functions $u : \mathbb{R}_+ \to \mathbb{R}$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $u'(\cdot) > 0 > u''(\cdot)$ and $h'(\cdot) > 0 > -h''(\cdot)$. Most of our statements and all of our analytical characterizations do, however, rely on the iso-elastic specification. In Section XX we discuss the case $\gamma = 1$.

cohort is

$$\nu(\theta, z^t) = u(c^y(\theta, z^t)) - \theta h(e(\theta, z^t)) + \beta \sum_{z^{t+1} \succ z^t} \pi(z^{t+1}|z^t) u(c^o(\theta, z^{t+1})).$$

Let $\nu(\hat{\theta}|\theta, z^t)$ be the utility of an agent θ, at a given history z^t, that announces to be of type $\hat{\theta}$,

$$\nu(\hat{\theta}|\theta, z^t) = u(c^y(\hat{\theta}, z^t)) - \theta h(e(\hat{\theta}, z^t)) + \beta \sum_{z^{t+1} \succ z^t} \pi(z^{t+1}|z^t) u(c^o(\hat{\theta}, z^{t+1})).$$

An allocation $(c^y(\theta, z^t), e(\theta, z^t), c^o(\theta, z^{t+1}))_{\theta \in \Theta, t, z^t}$ is ***incentive compatible*** if for all $\theta, \hat{\theta}, t, z^t$,

$$\nu(\theta, z^t) \geq \nu(\hat{\theta}|\theta, z^t) \tag{2}$$

A standard result whose proof we omit for brevity is that an allocation is incentive-compatible if and only if $\forall\ \theta, t, z^t$, we have $\dot{\nu}(\theta, z^t) = -h(e(\theta, z^t))$ and $\dot{e}(\theta, z^t) \leq 0$. In what follows we ignore the second-order monotonicity constraint and focus on a relaxed program, for which only the envelope condition is imposed.

2.1 The Planner's Program

At this point, it is important to explain the precise notion of risk sharing that we use. When agents are called to make decisions regarding the allocation of risk, only agents of their cohort are alive. Because preferences are state-independent, there is little meaning to the risk-sharing question if we do not define welfare from an ex-ante perspective. We follow Ball and Mankiw (2007); Gottardi and Kubler (2011) in taking this ex-ante perspective.[8] We assume that as a society we are concerned with the risk at the birth of future generations.

Consistent with the optimal risk-sharing idea, the planner solves a Utilitarian program,

$$\sum_t \sum_{z^t} \pi(z^t) \delta^t \int_\Theta \nu(\theta, z^t) \chi(\theta) d\theta. \tag{3}$$

[8] Ball and Mankiw (2007) consider the allocation that would result from complete Arrow-Debreu security markets if agents could trade at the beginning of times, only knowing when they would be born.

2.2 Risk Sharing

Because preferences are separable and iso-elastic there is perfect risk sharing if and only if consumption shares do not vary across states of nature.[9] That is, for $i = y, o$, let, $C^i(\theta, z^t) = \int_\Theta c^i(\theta, z^t) f(\theta) d\theta$, then we have the following definitions.

Definition 1. An allocation displays ***perfect risk-sharing within cohorts*** if $\forall t$, z^t, z, $\tilde{z}$ θ and $\hat{\theta}$,

$$\frac{u'(c^y(\theta, z^t, z))}{u'(c^y(\theta, z^t, \tilde{z}))} = \frac{u'(c^y(\hat{\theta}, z^t, z))}{u'(c^y(\hat{\theta}, z^t, \tilde{z}))} \quad \text{and} \quad \frac{u'(c^o(\theta, z^t, z))}{u'(c^o(\theta, z^t, \tilde{z}))} = \frac{u'(c^o(\hat{\theta}, z^t, z))}{u'(c^o(\hat{\theta}, z^t, \tilde{z}))} \tag{4}$$

With iso-elastic preferences, (4) is equivalent to there existing functions $s^i(\theta, z^{t-1})$, such that

$$\frac{c^i(\theta, z^t)}{C^i(z^t)} = s^i(\theta, z^{t-1}) \quad \forall \theta \in \Theta, \quad i \in \{y, o\}. \tag{5}$$

In most of what follows we will rely on (5) to assess perfect risk sharing.

Definition 1 refers to how resources are split between members of the same group. Given our focus on risk-sharing across groups of agents, it will be important to define perfect risk-sharing between workers and retirees.

Definition 2. An allocation displays ***perfect risk-sharing*** if for all t, $z^t \succ z^{t-1}$ both (4) and

$$\frac{u'(c^o(\theta, z^t))}{u'(c^y(\theta, z^t))} = \frac{\delta}{\beta}, \quad \forall \theta, z^t, \tag{6}$$

are satisfied.

Whereas (4) is a direct application of Borch's rule, equation (7) relies on some additional assumptions that we have made about the environment – $f(\theta)$ does not vary with z^t, t – and the planner's objective — $\chi(\theta)$ does not vary with z^t, t. If we further recall our use of iso-elastic preferences, this lead to a notion of risk sharing between workers and retirees represented by a type-by-type restriction of the following form. For all t, z^t there exist $a(\theta, z^{t-1})$ such that

$$\frac{c^o(\theta, z^t)}{c^y(\theta, z^t)} = a(\theta, z^{t-1}) \quad \forall \theta \in \Theta. \tag{7}$$

[9] For non-iso-elastic preferences, perfect risk-sharing means that the marginal rate of substitution does not vary between states: a standard Borch Rule.

In many circumstances we may be after a weaker notion of risk-sharing between groups, which obtain when we take the consumption ratio between retirees and workers as a function $\Omega : z^t \longrightarrow \mathbb{R}_+$ given by

$$\Omega(z^t) = \frac{C^o(z^t)}{C^y(z^t)}$$

and consider the following definition of perfect risk sharing.

Definition 3. An allocation displays ***perfect risk-sharing between workers and retirees*** if $\Omega(z^{t-1}, z) = \Omega(z^{t-1}, \overline{z})$ for all t, $z^{t-1}, \overline{z}$.

Of course, this weaker notion is implied by Definition 2, and, as we shall see, is not incompatible with an efficient distribution of resources between groups even when (7) is not satisfied.

The amount of effort different agents make in the different states of the world will also be relevant for determining the efficient use of all available resources. While we will have something to say about the issue, it will not be our main concern. Our definition of perfect risk sharing relates only to how consumption is split across agents.

2.3 First Best

It is useful to start with the first best program in which we ignore the incentive compatibility constraint (2). This allows us to provide a characterization for efficient allocations that will be useful as a benchmark.

To facilitate the comparison of the allocations derived herein with those in later Sections we allow the planner to assign different Pareto weights to different types. Because our program is concave we can in this case choose any allocation in the frontier of each generation's utility possibility set. We do sacrifice some generality by restricting Pareto weights on type-θ agents born in period t to be of the form $\delta^t \chi(\theta)$, with $\chi(\theta) > 0$, $\int_\Theta \chi(\theta) d\theta = 1$. Again, this restriction is not relevant for our purposes.

We thus define,

$$W_\chi(z^t) := \int_\Theta \nu(\theta, z^t)\chi(\theta) d\theta,$$

and write the planner's objective,

$$\sum_t \sum_{z^t} \pi(z^t) \delta^t W_\chi(z^t). \tag{8}$$

The first best problem is concave which permits characterization with standard methods. The first best allocation is

$$u'(c^y(\theta,z^t)) = \frac{f(\theta)}{\chi(\theta)}\overline{\eta}(z_t),\ u'(c^o(\theta,z^t)) = \frac{\delta}{\beta}\frac{f(\theta)}{\chi(\theta)}\overline{\eta}(z_t),\ h'(e(\theta,z^t)) = \frac{f(\theta)}{\theta\chi(\theta)}z_t\overline{\eta}(z_t)$$

with

$$\overline{\eta}(z_t) \propto z_t^{\frac{-\sigma\gamma}{\sigma+\gamma-1}},$$

i.e., it is characterized by perfect risk-sharing. Furthermore, $\Omega(z^t) = \left(\beta/\delta\right)^{1/\sigma}$, which depends neither on current nor on past aggregate shocks. Indeed, with iso-elastic preferences, optimal risk-sharing corresponds to having agents consume a fixed share of aggregate consumption in all states of the world, at all nodes of a society's history.

Definition 4. An allocation is ***devoid of memory*** if it is of the form:

$$(c^y(\theta,z^t), e(\theta,z^t), c^o(\theta,z^{t+1})) = (c^y(\theta,z_t), e(\theta,z_t), c^o(\theta,z_{t+1})), \quad \forall\theta\in\Theta$$

In other words, we define an allocation as being devoid of memory if it does not depend on past *aggregate* history. The first-best allocation in an endowment economy with proportional government expenditures is an example of an economy characterized by perfect risk-sharing and the absence of memory.

Investments create history dependence in a first-best economy. So we will often return to the endowment economy to highlight the consequences of incentive provision.

2.4 Constrained Efficiency

The constrained efficient program maximizes the planners' objective (32) subject to the resource constraints (33), and the incentive compatibility constraints (2). Towards a characterization, note that we can use the current aggregate shock, x, and the aggregate consumption promises for retirees, τ, as state variables to rewrite the problem recursively as

$$v(\tau,x) = \max\int_\Theta \left[u(c^y(\theta)) - \theta h(e(\theta)) + \beta\mathbb{E}_z u(c^o(\theta,z'))\right] f(\theta)d\theta \\ + \delta\mathbb{E}_z v(\tau'(z'),z'),$$

subject to

$$\int_{\Theta}(c^y(\theta)+\tau)f(\theta)d\theta \leq z\int_{\Theta} e(\theta)f(\theta)d\theta,$$

$$\tau'(z') \geq \int_{\Theta} c^o(\theta,z')f(\theta)d\theta,$$

and

$$\dot{\nu}(\theta) = -h(e(\theta)).$$

Because the program – henceforth, Program $\mathbb{P}_0$ – is concave standard techniques apply. The following proposition is immediate from the program's first-order conditions.

Proposition 1. *Constrained efficient allocations satisfy*

$$\mathbb{E}_t\left[\frac{1}{u'(c^y(\theta,z^t))}\right] = \frac{\delta}{\beta}\mathbb{E}_t\left[\frac{1}{u'(c^o(\theta,z^t))}\right], \tag{9}$$

$$\frac{u'(c^y(\theta,z^t))}{u'(c^y(\hat{\theta},z^t))} = \frac{u'(c^o(\theta,z^{t+1}))}{u'(c^o(\hat{\theta},z^{t+1}))}, \quad \forall \theta,\hat{\theta},z^t,z^{t+1}\succ z^t. \tag{10}$$

The first part of Proposition 1 states that the marginal value of resources to retirees and workers are equalized at any period and any state of nature. The second part of Proposition 1 has some immediate consequences. First, since the ratio on the left-hand side of (10) does not depend on z^{t+1} neither does the ratio on the right-hand side, which implies optimal risk-sharing among retirees.

Corollary 1.1. *At the constrained optimum there is perfect risk-sharing among retirees.*

Workers are provided incentives by both higher current consumption and higher future expected utility. They are indifferent about how the expected utility is delivered tomorrow, so the planner does it at the lowest possible cost, which means by sharing aggregate risk optimally among retirees.

Define the ***replacement ratio*** for a type θ in aggregate state z^{t+1} as the ratio between θ's consumption as a retiree in state $z^{t+1}\succ z^t$, $c^o(\theta,z^{t+1})$, and her consumption as a worker, $c^y(\theta,z^t)$. Note that from the perspective of the period, t the replacement ratio is a random variable.

Corollary 1.2. *At the constrained optimum, for all z^{t+1}, the replacement ratio is independent of θ,*

$$\frac{c^o(\theta,z^{t+1})}{c^y(\theta,z^t)} = \frac{C^o(z^{t+1})}{C^y(z^t)}, \quad \forall \theta,t,z^{t+1}\succ z^t.$$

Equivalently,

$$s(\theta, z^t) = \frac{c^y(\theta, z^t)}{C^y(z^t)} = \frac{c^o(\theta, z^{t+1})}{C^o(z^{t+1})}, \quad \forall \theta, t, z^{t+1} \succ z^t. \tag{11}$$

Efficiency, therefore, requires the shares of an agent's cohort's consumption to be the same when the agent is a worker and when he or she retires. Hence, in all that follows we use the following definition.

Definition 5. The ***economy's replacement ratio*** in z^{t+1} is $R(z^{t+1}) := C^o(z^{t+1})/C^y(z^t)$.

Finally, if retirees' consumption does not depend on past aggregate shock, it is because incentive provision through retirement benefits was invariant across aggregate states when these agents were working. The following corollary, which is another immediate consequence of (10) shows what it teaches us about risk sharing among these retirees when they were workers.

Corollary 1.3. *If at the constrained optimum the consumption of retirees in period $t+1$ is devoid of memory, then there is perfect risk-sharing among workers in period t.*

3 Risk-sharing, Incentive Provision and Backloading

We start our analysis by showing how incentive provision relates to risk sharing.

3.1 Risk-sharing within and between cohorts

From Proposition 1, recall that the aggregate marginal value between workers and retirees is equalized. Using the iso-elastic property on (10), we have

$$\begin{aligned} \mathbb{E}_t\left[c^y(\theta, z^t)^\sigma\right] &= \frac{\delta}{\beta}\mathbb{E}_t\left[c^o(\theta, z^t)^\sigma\right], \\ \mathbb{E}_t\left[s^y(\theta, z^t)^\sigma\right] C^y(z^t)^\sigma &= \frac{\delta}{\beta}\mathbb{E}_t\left[s^o(\theta, z^t)^\sigma\right] C^o(z^t)^\sigma, \quad \text{and,} \\ M_y^\sigma(z^t) C^y(z^t)^\sigma &= \frac{\delta}{\beta} M_o^\sigma(z^t) C^o(z^t)^\sigma \end{aligned} \tag{12}$$

Using $\Omega = C^o/C^y$, we obtain the following expression which connects risk-sharing between groups with the cross-sectional consumption dispersion within groups.

Proposition 2. *If an allocation is constrained efficient, then*

$$\left(\frac{\beta}{\delta}\right)^{1/\sigma}\left[\frac{M_y^\sigma(z^t)}{M_o^\sigma(z^t)}\right]^{\frac{1}{\sigma}} = \Omega(z^t), \tag{13}$$

where $M_y^\sigma(z^t) = \int_\Theta s^y(\theta, z^t)^\sigma f(\theta)d\theta$ *for* $s^y(\theta, z^t) = c(\theta, z^t)/C^y(z^t)$*, with analogous definition for* $M_o^\sigma(z^t)$*.*

Recall that $\left(\beta/\delta\right)^{1/\sigma}$ is the first-best ratio of total retirees' and workers' consumption while $M_y^\sigma(z^t)$ is a measure of dispersion for the consumption of workers and $M_o^\sigma(z^t)$ the analogous statistics for retirees. The independence of $\Omega(z^t)$ concerning the current aggregate shock z_t is a weaker notion of optimal risk-sharing between cohorts since we only require perfect risk-sharing on aggregate measures, not agent-by-agent. It is nonetheless a relevant notion for the question we posed in the title of this paper. Condition (13) tell us that this weaker form of optimal risk-sharing is directly related to optimal risk-sharing within cohorts, exactly as found by Demange (2008).

According to Corollary 1.1, the share, $s^o(\theta, z^t)$, to which a type θ retiree is entitled does not vary across z_ts. Using (10), we obtain that $M_o^\sigma(z^t) = M_y^\sigma(z^{t-1})$. So the behavior of $\Omega(z^t)$ only depends on how $M_y^\sigma(z^t)$ varies across states. Optimal risk sharing between the two groups in the weak (aggregate) sense requires $\Omega(z^t)$ to be independent of z_t.

Corollary 2.1. $\Omega(z^t)$ *is invariant to* z_t *if and only if for all* z^{t-1}*,* z*,* $\tilde{z}$*,* $M_y^\sigma(z^{t-1}, z) = M_y^\sigma(z^{t-1}, \tilde{z})$*.*

Since $M_y^\sigma(z^t) \approx 1 + 0.5\sigma(\sigma - 1)\,\mathrm{Var}\left(s^y(\theta, z^t)\right)$ it is clear that the way the dispersion of consumption shares varies across aggregate states affects $\Omega(z^t)$ differently depending on whether $\sigma \lesseqgtr 1$.[10] In particular, note that if agents have log preferences, then $\sigma = 1$ and $M_y^\sigma(z^t) = M_o^\sigma(z^t) = 1$: there is perfect risk sharing between workers and retirees. How about perfect risk sharing in the strong (individual) sense? Proposition 3, below, shows that this is also the case.[11]

Proposition 3. *The constrained efficient allocation displays perfect risk-sharing if and only if it displays perfect risk-sharing among workers.*

[10] Of course this is immediate from Jensen's inequality. The approximation is useful to provide a sense of its quantitative relevance if one has information on the variance.

[11] It is not, however, a consequence of Proposition 2, and is proven in the appendix.

The relationship between incentive provision and risk-sharing across groups was, to the best of our knowledge, first investigated by Demange (2008). She compared two groups, one which was subject to moral hazard and another which was not. To induce agents to make effort, consumption was made dependent on the agent's output thus creating consumption dispersion. Dispersion in consumption affects the marginal value of resources in the hands of the planner since any resources handed to the planner must be returned to the agents in an incentive-compatible fashion.[12] Now optimal risk-sharing is all about making sure that the marginal rate of substitution between resources at any two states of nature is equalized among agents, or, in her case, groups of agents.

Under identical iso-elastic preferences consumption shares of retirees are invariant to the state of nature. If the same is true for workers, then the amount of dispersion only determines the relative share of the two groups, but perfect risk sharing is optimal. However, to the extent that consumption dispersion, hence, incentive provision, varies across states of nature, so does the marginal value of resources for the group for which incentives must be provided. In this case, the share each group would optimally obtain varies across states of nature: perfect risk sharing ceases to be optimal.

3.2 Backloading Incentives through Social Security

The forces Demange (2008) describes are clearly at play in the findings of Section 3.1. However, applying her findings to the workers'/retirees' risk-sharing problem is somewhat more involved. Although retirees need not be incentivized, their expected consumption is an integral part of the incentive provision problem that they were given in their working years.[13] Hence, to understand risk-sharing across workers and retirees, it is important to take into account a second aspect, the backloading of incentives.

With a concave utility, incentives are more efficiently provided by spreading consumption differences across periods. Inequality between retirees, therefore, characterizes any constrained efficient allocation: at the optimum, it is always the case that retirement benefits depend on each agent's relative (to his cohort) past earnings. What we ask here is whether it should also depend on past *aggregate* history, and in particular, how this dependence relates to optimal risk-sharing during working years.

As it turns out, a very straightforward understanding of backloading is made possible

[12]The precise way in which dispersion affects the marginal value of resources in the planner's hand is what M_y^σ and M_o^σ measure.

[13]Bastani et al. (2018), for example, explore the use of pensions to implement history-dependent taxation.

by our using a one-period representation of this dynamic incentive problem.

A one period interpretation Define the following *backloading function*, z^{t+1},

$$\Phi(z^t) := 1 + \beta \mathbb{E}\left[Q_t(z^{t+1}) \frac{C^o(z^{t+1})}{C^y(z^t)} \middle| z^t \right], \tag{14}$$

where $Q_t(z^{t+1}) = \beta \left(C^o(z^{t+1})/C^y(z^t)\right)^{-\sigma}$. Recalling that $R(z^{t+1})$ is the replacement ratio at history z^{t+1}, $\Phi(z^t) = 1 + \beta \mathbb{E}\left[R(z^{t+1})^{1-\sigma} | \, z^t \right]$

Proposition 4. *The utility attained by a θ type agent born into history z^t at the constrained optimum is*

$$\nu(\theta, z^t) = \Phi(z^t) \left[u(c^y(\theta, z^t)) - \frac{\theta}{\Phi(z^t)} h(e(\theta, z^t)) \right], \tag{15}$$

Interestingly for our purposes, equation (15) allows one to re-interpret the problem as that of an economy in which agents live only one period, but whose heterogeneity is now state-dependent and given by $\theta/\Phi(z^t)$. This has important consequences for efficient labor wedges.

Indeed, to provide incentives, compensation must be given in the form of more consumption for those who produce more. This is, however, costly if the utility function is concave in consumption. Second-best principles teach us that it is optimal to spread distortions across all margins. In particular, distortions in the leisure/consumption margin should be also introduced to relax incentive constraints and alleviate the costs associated with consumption dispersion. The magnitude of the labor wedge, i.e., the difference between the marginal rate of substitution between effort and consumption and the marginal rate of transformation between the two – $\tau^y : \Theta \times Z^t \longrightarrow [0, 1) \quad \forall i, t$, such that

$$\tau^y(\theta, z^t) = 1 - \frac{\theta h'(e(\theta, z^t))}{z_t u'(c^y(\theta, z^t))}, \tag{16}$$

offers an alternative signal of how hard the provision of incentives is.

If redistribution requires positive marginal tax rates, then increasing Φ relaxes incentive constraints and allows wedges to be reduced. The intuition is easier in the case of discrete types. If only downward – in the sense of high productivity (low θ) agents envying low productivity – constraints bind, then increases in Φ reduce the relative disutility‘ differences from the different outputs that agents must produce. This relaxes incentive constraints.[14] With a continuum of agents the argument is more involved since incentive compatibility is violated if one increases Φ but does not change the allocation.[15]

In the next proposition, we consider the following thought experiment. Assume that in some state z^t, $\Phi(z^t)$ is increased by some exogenous reason, i.e., without any change in the allocation.[16] We then show that, if we start from an allocation displaying positive labor wedges, then this exogenous increase in backloading, $\Phi(z^t)$, allows us to increase the utility of all members of the current generation while holding the allocations of all other generations and at all nodes of history fixed. The new allocation is characterized by lower marginal tax rates and a lower value for the moment condition $M_y^\sigma(z^t)$.

Proposition 5. *Let $(u^y(\theta, z^t), h(\theta, z^t), (u^o(z^{t+1}))_{z^{t+1}\succ z^t})_{\theta,t,z^t}$ be an allocation satisfying (11) and characterized by non-negative labor wedges. Let $\Phi(z^t)$ be the backloading terms. If at node $\bar{z}^t$ there is an unexpected exogenous small increase, $d\Phi$, in $\Phi(\bar{z}^t)$, then it is possible to construct a new incentive-feasible allocation*

$$(\hat{u}^y(\theta, z^t), \hat{h}(\theta, z^t), (\hat{u}^o(z^{t+1}))_{z^{t+1}\succ z^t})_{\theta,t,z^t}$$

such that

i) for all θ, z^t, t, $\hat{u}^o(\theta, z^t) = u^o(\theta, z^t)$,

ii) for all θ, t, $z^t \neq \bar{z}^t$, $\hat{u}^y(\theta, z^t) = u^y(\theta, z^t)$, $\hat{h}(\theta, z^t) = h(\theta, z^t)$, and

iii) for all θ, $\hat{u}(\theta, \bar{z}^t) - \theta\hat{h}(\theta, \bar{z}^t) > u(\theta, \bar{z}^t) - \theta h(\theta, \bar{z}^t)$.

Moreover, labor wedges, $\tau^y(\theta, \bar{z}^t)$, are lower and $M_\sigma^y(\bar{z}^t)$ is smaller at the new allocation.

The last part of Proposition 5 shows that the distortions required for incentive provision as captured by $\tau^y(\theta, \bar{z}^t)$ are reduced by the exogenous increase in Φ. There is a lower dispersion in utility flows from consumption and a lower cross-sectional moment, $M_\sigma^y(\bar{z}^t)$, again pointing to the fact that incentive constraints were relaxed. It is also the case – see the proof of Proposition 5 – that $(\hat{u}(\theta, \bar{z}^t), \hat{h}(\theta, \bar{z}^t)) > (u(\theta, \bar{z}^t), h(\theta, \bar{z}^t))$ for all θ, where the vector inequality implies that neither entry is smaller and at least one is strictly greater at the new allocation.

The purpose of the thought experiment was to isolate the impact of changes in Φ on the welfare of a single generation. The reform that was made possible by this change

[14] With a finite number of types, the upward constraints are slack in the Utilitarian program. We need not worry about violating them if the increase in Φ is not too large.

[15] E.g., the planner would like to attain $\dot{v} \geq 0$ but incentive compatibility requires $\dot{v} = -h(\theta)/\Phi < 0$.

[16] One could imagine, for example, exogenous changes in agents' beliefs.

increased everyone's utilities by the same amount, leading to negative marginal tax rates for the most productive agent, which is not optimal for a Utilitarian planner. Along the same lines, while the allocation described in the proposition is Pareto superior, it was only the generation born at history node z^t that benefited. Yet, a lower $M_\sigma^y(z^t)$ implies that transferring resources from workers to retirees would increase the value of the planners' objective. Proposition 5 aimed not at providing a characterization of the optimum, but simply at highlighting how the backloading of incentives can be used to relax constraints and improve allocations.

The amount of backloading trades-off this relaxation in incentive constraints with with changes in utility for the next generation. If backloading is optimally used, then, at the margin, its benefits equal its costs. Finding the allocation under which this equality occurs would, in principle, allow us to characterize the optimum. Unfortunately, finding such an allocation is far from trivial. So, in Section 3.3, instead of following this path, we assume that there is perfect risk-sharing at the optimum, fully characterize the allocation, and offer conditions on the primitives for this to be the case. Before, however, we discuss the generality of our results with regard to the technology.

Generalizing the Technology We have focused on an endowment economy with no government consumption. Yet, except for the characterization of first-best allocations, all results thus far remain valid if we considered instead a general technology, $Y_t = zF(N_t, K_{t-1})$, where N_t is the aggregate supply of efficiency hours in period t, K_{t-1} is the capital stock and z is the current period productivity shock. Note that, with this technology, an allocation is feasible if for every period, t, and every history, z^t,

$$\int_\Theta \left\{c^y(\theta, z^t) + c^o(\theta, z^t)\right\} f(\theta) d\theta \leq z_t F\left(\int_\Theta e(\theta, z^t) f(\theta) d\theta, K_{t-1}\right) - K_t - G_t.$$

3.3 Efficiency and Perfect Risk-Sharing

Memory, in the sense of dependence on aggregate history, has been shown to characterize efficient allocations in environments that are very similar to ours – da Costa and Farinha Luz (2018); Veracierto (2020). This makes any attempt at a general characterization a futile exercise. So, instead of taking this route, in this section, we proceed as follows. First, we show that if we impose perfect risk sharing, then an analytical solution for the backloading term, $\Phi(z^t)$, obtains. We can use this expression to check the restrictions in the primitives under which no violation of optimality conditions is generated. This allows us to derive the necessary and sufficient conditions for perfect risk sharing to be optimal. As we shall see, these conditions are very restrictive. So, we consider a doubly restricted problem in which allocations are required to display perfect risk sharing and history independence. The variation in incentive provision across states is fully accomplished by state-dependent labor wedges.

It is only in Section 4, however, that we (numerically) find the optimal allocation when the assumptions that lead to perfect risk-sharing being optimal are relaxed. We use some of the insights from the analysis in this section to provide an intuition for our quantitative findings.

Perfect Risk-sharing With iso-elastic preferences, perfect risk-sharing implies that the share of each agent's consumption does not vary across states. It is intuitive and indeed true that, in this scenario, the share of effort that each agent must exert will also be invariant across states. Furthermore, if incentives do not vary, then the central planner can - and will - choose the optimal correlation between consumption and current shocks.[17]

Proposition 6. *If a constrained efficient allocation displays perfect risk-sharing, then there exist functions $\phi^i : \Theta \times Z^{t-1} \to \mathbb{R}_+$, $i \in \{y, o, e\}$, such that:*

$$\begin{aligned} u'(c^y(\theta, z^t)) &= \phi^y(\theta, z^{t-1})\overline{\eta}(z_t), \\ u'(c^o(\theta, z^t)) &= \phi^o(\theta, z^{t-1})\overline{\eta}(z_t), \\ \theta h'(e(\theta, z^t)) &= \phi^e(\theta, z^{t-1}) z_t \overline{\eta}(z_t), \end{aligned}$$

where $\overline{\eta}(z_t) \propto z_t^{\frac{-\sigma\gamma}{\sigma+\gamma-1}}$.

The first two terms are almost by definition of perfect risk-sharing. Proposition 6 derives its content from the last equality and the fact that the multiplicative term $\overline{\eta}(z_t)$ has a closed-form expression that coincides with that derived for the first-best allocation. This has an immediate consequence for the cyclical behavior of labor wedges.

Corollary 6.1. *A constrained optimal allocation,*

i) displays perfect risk sharing only if $\tau(\theta, z^{t-1}, z) = \tau(\theta, z^{t-1}, z')$ for all θ, z^{t-1}, z, z', and;

[17] With general preferences, we can prove that this correlation is positive - the direction of the correlation is the same as the one that occurs at the first-best. Moreover, $\overline{\eta}(z^t)$ will be a decreasing function

ii) is devoid of memory only if for all θ, z^t, $\tau(\theta, z^t) = \bar{\tau}(\theta)$ for some $\bar{\tau}(\cdot)$.

Recalling that labor wedges are a meaningful statistic for incentive provision costs, we note that it is exactly when this statistic is invariant across states of nature that we have perfect risk-sharing at the optimum.

Another consequence of Proposition 6 is a restriction on the behavior of $\Phi(z^t)$ when the optimal allocation displays perfect risk-sharing.

Corollary 6.2. *There is perfect risk sharing at the constrained optimum only if $\Phi(z^t)$ does not depend on z_t.*

This last corollary establishes a connection between risk sharing and backloading which we further explore for the rest of this section. To do it we start by noting that, whenever $\Phi(z^t)$ is independent of z_t, the incentive provision associated with the one-period problem defined in Proposition 4 is invariant to z_t. It is then possible to show that its solution has the separable structure of Proposition 6.

Because shares are invariant across states of nature so is $\Omega(z^t)$, and we have perfect risk sharing between workers and retirees. In other words, for the endowment technology with iso-elastic preferences, it is only through variations in backloading across states of nature that incentive provision may vary with the states of nature. Next, we discuss when and how this happens.

Perfect Risk-Sharing and Backloading According to Lemma 3, in the appendix, if a constrained efficient allocation is devoid of memory, then $\Phi(z^t) = \hat{\Phi}(z^t)$, for

$$\hat{\Phi}(z^t) := 1 + \beta \left(\frac{\beta}{\delta}\right)^{\frac{1-\sigma}{\sigma}} \mathbb{E}\left[\left(\frac{z_{t+1}}{z_t}\right)^{\frac{\gamma[1-\sigma]}{\sigma+\gamma-1}} \Big| z^t\right]. \tag{17}$$

According to Corollary 1.3, an efficient allocation that does not display memory of aggregate shocks, displays perfect risk-sharing, while, according to Corollary 6.2, perfect risk-sharing requires $\Phi(z^t)$ to be independent of z^t. Combining these two results with Lemma 3 the following proposition obtains.

Proposition 7. *A constrained efficient allocation is devoid of memory if and only if, $\forall t, z^t$, $\Phi(z^t) = \hat{\Phi}(z^t)$, and $\hat{\Phi}(z^t)$ is independent of z^t.*

If $\Phi(z^t)$ is independent of z^t, then it is immediate from Proposition 4 that incentive constraints are invariant to the state of the economy. If the moment condition is invariant, then, perfect risk-sharing obtains.

The next two corollaries of Proposition 7 provide rather knife-edge conditions for the absence of memory: log utility or i.i.d. productivity growth.[18]

Corollary 7.1. *If* $\sigma = 1$ *the constrained optimal allocation is devoid of memory and displays perfect risk-sharing.*

Corollary 7.2. *If* z_{t+1}/z_t *follows an i.i.d. process then the constrained efficient allocation is devoid of memory and displays perfect risk-sharing.*

3.3.1 A Restricted Planner's Program

We have seen that perfect risk sharing does not characterize constrained efficient allocations in general. In this section we impose perfect risk sharing and characterize these third best allocations. We constrain the planner's choice by requiring the individual consumption shares $s(\theta, z)$ to be invariant to the aggregate state — $s(\theta, z) = s(\theta)$, with some abuse, and by requiring that the total consumption of workers and retirees to be history-independent. Let, $\Psi(z) := C^y(z)^\sigma \Phi(z)$.

To facilitate communication we index the aggregate states $j = 1, ...$ with $z_j > z_{j-1}$.

The planner's restricted program is, in this case,

$$\max_{(\Psi(z), C^o(z))_z, (v(\theta,z), h(\theta,z))_{\theta,z}, (s(\theta))_\theta} \sum_j \pi(j) \int_\Theta v(\theta, z) f(\theta) d\theta$$

subject to

$$v(\theta, j) = \Psi(j) \frac{s(\theta)^{1-\sigma}}{1-\sigma} - \theta h(\theta, j),$$

$$\int s(\theta) f(\theta) d\theta = 1,$$

$$v(\theta, j) = v(\underline{\theta}, j) - \int_{\underline{\theta}}^{\theta} h(r, j) dr,$$

and

$$\left[\Psi(j) - \beta \sum_k \pi(k|j) C^o(k)^{1-\sigma} \right]^{\frac{1}{1-\sigma}} + \beta C^o(j) \leq z_j \int_\Theta N\left(h(\theta, j)\right) f(\theta) d\theta.$$

[18] With log utility, dividing consumption-effort by z, the aggregate shock disappears from constraints and only affects utility through an additive term. The normalized variables do not depend on the aggregate shock, and an optimal allocation without memory obtains.

Solving this 'static' Mirrlees' model we find that

$$\frac{\tilde{\tau}(\theta)}{1-\tilde{\tau}(\theta)} = \left\{\mathbb{E}\left[s(r)^{\sigma}|r \leq \theta\right] - \mathbb{E}\left[s(\theta)^{\sigma}\right]\right\} \frac{F(\theta)}{f(\theta)\theta} s(\theta)^{-\sigma}$$

where

$$\tilde{\tau}(\theta) = 1 - \frac{C^y(j)^{\sigma} s(\theta)^{\sigma}\theta}{\Psi(j) z_j N'(h(\theta, j))}.$$

Note that these wedges use *backloading-adjusted productivities*, $\theta/\Psi(j)$.[19]. The main finding is that these 'adjusted wedges' do not vary across states of the world. Since actual wedges, $\tau(\theta, j)$, are defined as $1 - [C^y(j)^{\sigma} s(\theta)^{\sigma}\theta]/[z_j N'(h(\theta, j))]$, we have, $\tau(\theta, j) = \Psi(j)[\tilde{\tau}(\theta) - 1] - 1$, which decrease with $\Psi(j)$ for $\tilde{\tau} < 1$. Again we see how more backloading leads to lower wedges, even at this third best allocation.

As for $\Psi(j)$, the optimum is given by

$$\Psi(j) = \frac{\lambda_j C^y(j)^{\sigma}}{\pi(j)} \int_{\underline{\theta}}^{\bar{\theta}} s(\theta)^{\sigma} f(\theta) d\theta,$$

which we can show to imply

$$\frac{\Psi(j)}{\Psi(k)} = \frac{C^y(j)^{\sigma}/C^o(j)^{\sigma}}{C^y(k)^{\sigma}/C^o(k)^{\sigma}}.$$

Note that $\Psi(z)$ relates the consumption of current workers with that of these workers after they retire, while the right-hand side relates the consumption of current workers and current retirees.

3.4 The $\gamma = 1$ Case

An interesting perfect risk sharing result arises for the case $\gamma = 1$ – see Appendix F. This case is special since the absence of memory and perfect risk sharing arise despite the fact that

$$\hat{\Phi}(z^t) := 1 + \beta \left(\frac{\beta}{\delta}\right)^{\frac{1-\sigma}{\sigma}} \mathbb{E}\left[\left(\frac{z_{t+1}}{z_t}\right)^{\frac{1-\sigma}{\sigma}} \Big| z^t\right]$$

will typically vary with z.

This finding is in contrast with the statement of Proposition 7, which is valid under the assumption that $\gamma > 1$. So, what is going on? With $\gamma = 1$, any variation in the

[19] To be precise, the backloading term is $\Phi(j) = \Psi(j) C^y(j)^{-\sigma}$.

incentive requirements due to changes in $\Phi(z)$ are efficiently adjusted by changes in effort with no changes in consumption.[20]

4 Numeric Results

In this section, we numerically implement our model. The purpose of these exercises is twofold. First, we substantiate the theoretical findings by showing how each prediction arises in a stylized version of our economy. Second, we rely on parameter values typically used in the literature to assess the quantitative relevance of the forces that are isolated in the theory section.

For the numeric illustration, we assume that $\theta \sim U[0.5, 1.5]$, $\sigma = \gamma = 2$ and that the planner discounts future generations at the same rate that each individual discounts the future: $\delta = \beta = 0.95^{20}$ and $\chi(\theta) = f(\theta)$. Finally, we assume that the log of aggregate shock z_t follows an $AR(1)$ process:

$$\ln z_t = \rho \ln z_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, (0.1)^2)$$

Note that this process implies

$$\frac{z_{t+1}}{z_t} = z_t^{\rho-1} e^{\epsilon_{t+1}} \Rightarrow \hat{\Phi}(z^t) = 1 + B\eta(z_t)^{1-\rho},$$

where $B > 0$ is some deterministic constant and $\eta(z) \equiv z^{\frac{\gamma}{\sigma+\gamma-1}} = z^{2/3}$.

We choose this stochastic process because, by corollary 7.2, this implies that the optimal allocation will display perfect risk-sharing if and only if $\rho = 1$. Furthermore, this process is mean-reverting for $\rho < 1$. To strengthen the mean-reverting force, from now we consider the case where the aggregate shock is i.i.d in levels: $\rho = 0$.

Let us pause for a moment and think about what we expect to find. Assume (as will indeed occur) that our 'true' Φ function - which depends on the endogenous consumption growth rate - behaves in a way similar to $\hat{\Phi}$ - which is an exogenous and increasing function; then we expect the backloading function - $\Phi(z^t)$ - to increase in good times and decrease in bad ones. As a consequence (recall proposition 5 and the discussion that preceded it), both the labor wedge - $\tau^y(z^t)$, and the consumption dispersion - M_y - should be lower in good times. Under these conditions, Proposition 2 predicts a smaller

[20] Indeed, the multiplier associated with the incentive compatibility constraint is independent of z – see equation (42) – in the appendix.

share of resources entitled to retirees - $\Omega(z^t)$.[21]

Figures 1 - 4 display these four functions Φ, τ^y, M_y and Ω.[22] Even though these functions are aligned with our analytical local results, τ^y, M_y, and Ω are close to constant functions. This suggests that the optimal allocation approximately displays perfect risk-sharing.[23]

To further investigate our finding, we compute the ratio between the true backloading function - $\Phi(z^t)$ - and the one that arises if we try to implement an allocation that is devoid of memory and displays perfect risk-sharing - $\hat{\Phi}(z^t)$. As Figure 5 displays, these two 'backloading' functions are roughly the same.

Second, under perfect risk-sharing, aggregate consumption is proportional to $\eta(z_t)$. In Figure 6, we compute the ratio between the actual aggregate consumption - $C^o(z^t)$ - and $\eta(z_t)$ and show that it's approximately constant.[24]

Finally, recall that since $\rho = 0$, our process is i.i.d. in levels. Thus, we can compute the correlation of aggregate consumption $C^o(z^t)$ with lagged $\eta(z_t)$. The absence of memory would imply that this correlation should be zero for any lag different than zero, which is approximately true as displayed in Figure 7.

Our numerical evaluation seems to be finding an absence of memory where theory says there should be. What is going on? As in Veracierto (2020), deviations from optimal risk-sharing are so small that the slackness in the incentive constraints is not distinguished from zero from a numeric perspective.[25] The allocation is (approximately) devoid of memory, thus (approximately) displaying perfect risk-sharing. Does it mean that we should simply disregard the possibility of memory and focus on perfect risk-sharing allocations?

To focus on the aspects of our formulation that may be driving our numeric findings, recall the term

$$\Phi(z^t) = 1 + \frac{1}{C^y(z^t)}\mathbb{E}\left[Q_t(z^{t+1})C^o(z^{t+1}) \middle| z^t \right].$$

It defines the present value of generation t (born into state z^t) consumption. This present value is risk-adjusted since future consumption is priced by the stochastic discount factor (SDF), $Q_t(z^{t+1}) = \beta \left(C^o(z^{t+1})/C^y(z^t)\right)^{-\sigma}$. A well-known fact about this SDF is that it does not display enough variability to 'price' the equity term, $C^o(z^{t+1})$. Not sur-

[21] As usual, the labor wedge for the high type - $\underline{\theta}$ is always zero.

[22] For figure 2, we displayed for $\theta = 1$, the other interior types are also decreasing functions.

[23] The numerical finding that optimal allocations are close to the absence of memory is also present in Veracierto (2020).

[24] We would expect large fluctuations if memory played a relevant role.

[25] See discussion before Proposition 5.

prisingly, the behavior of the backloading term does not display enough variability with respect to the aggregate state. Indeed, consider the expected return $\overline{R}_e$ of an asset that pays the aggregate retirement consumption tomorrow - the relevant notion of 'equity' in this economy.[26] If we compare it with a risk-free asset $R_f \equiv 1/\mathbb{E}[Q]$, the annualized risk-premium $\overline{R}_e^{1/20} - R_f^{1/20}$ is only 0.44% in this numerical example, while the equity-premium range, found in the literature is $5 - 7\%$.

These numbers are not perfectly comparable, but it provides a useful guide to understanding why the model is failing to deliver allocations with memory, namely the absence of enough volatility in the stochastic discount factor Q_t. To assess whether this effect lies behind the small quantitative impacts we find in our numeric explorations, we consider a variation of our model that is reverse-engineered to price equity.

4.1 Taste Shocks

We expand our model to consider aggregate taste shocks. That is, we assume that

$$\mathscr{U}(c^y, c^o, e) = u(c^y) - \theta h(e) + \xi\beta u(c^o),$$

where ξ is a random variable taste shock, that hits every one of a given cohort.

We adopt the following notation. Let ξ_t denote the common taste shock at period t, and $\xi^t = (\xi_1, ..., \xi_t)$, the history of taste shocks up to period t. Let $x_t \equiv (z_t, \xi_t)$ and $x^t \equiv (x_1, ..., x_t)$ be the aggregate history of shocks up to time t. We use $\pi(x^t)$ to denote the probability of history t, understanding that for every t there are a finite number of possible histories, x^t.

Consistent with the optimal risk-sharing idea, the planner solves a Utilitarian program,

$$\sum_t \sum_{z^t} \pi(z^t)\delta^t \Pi_{i=0}^t \xi_i \int_\Theta \nu(\theta, z^t)\chi(\theta)d\theta, \tag{18}$$

where we normalize $\xi_0 = 1$.

By adopting the same procedure we used for deriving (14) we obtain also for this model,

$$\Phi(x^t) = 1 + \frac{1}{C^y(x^t)}\mathbb{E}\left[Q_t(x^{t+1})C^o(x^{t+1})\middle|\, x^t\right], \tag{19}$$

but now for $Q_t(x^{t+1}) = \beta\xi_{t+1}\left(C^o(x^{t+1})/C^y(x^t)\right)^{-\sigma}$.

[26] Formally, $P_e(z^t) \equiv \mathbb{E}_t\left[Q(z^t)C^o(z^{t+1})\right]$, $R_e(z^{t+1}) \equiv C^o(z^{t+1})/P_e(z^t)$ and $\overline{R}_e \equiv \mathbb{E}[R_e]$.

To better illustrate how a volatile enough stochastic discount factor is important to (numerically) break the absence of memory, we turn off the productivity shock: $z_t = 1$ $\forall t$ and assume that the log of ξ_t follows an $AR(1)$ process:

$$\ln \xi_t = 0.7 \ln \xi_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, (0.7)^2) \Rightarrow \hat{\Phi}(x^t) = 1 + B\xi_t^{0.7}$$

where $B > 0$ is some deterministic constant. In the online appendix, we prove that, if the allocation is devoid of memory, then it does not depend on any ξ_t. In particular, the correlation of the retiree's consumption $C^o(x^t)$ against any k-lagged SDF/taste shock ξ_{t-k} should be zero. In Figure 8, we plot this correlation, which is far from zero for the initial lags. Thus, the optimal allocation now will (numerically) have memory.

5 Implementation

As we know, there are multiple ways of decentralizing an implementable allocation. The most natural one is, perhaps, under a pay-as-you-go public social security system. A worker with earnings $z_t e(\theta, z^t)$ makes a compulsory contribution $z_t e(\theta, z^t) - c^y(\theta, z^t)$ to the Social Security system and is entitled to a conditional (on the aggregate state) benefit $\left(c^o(\theta, z^t, z)\right)_z$.

We have, up to this point, not referred to any form of asset trade. Indeed there are no real assets in this economy since the only technology available has constant returns to scale. A natural possibility that arises in this type of setting is public debt. That is, the government issues a nominal asset that promises payment in the next period financed through taxes raised from the next generation. For this to work, the return on the public debt must be state-contingent.

Another possibility is to allow the representative firm in the economy to sell stocks. As we have discussed the gross profits are always zero, so, unless there is a bubble in the value of this asset it will not serve to transfer resources intertemporally. What we shall consider is a tax on this asset that makes after-tax profits non-zero.[27] Assume that a type θ agent facing aggregate shock z_t and history z^t, chooses a vector of consumption-effort, $\left(c^y(\theta, z^t), \left(c^o(\theta, z^{t+1})\right)_{z^{t+1}}, e(\theta, z^t)\right)$, and shares of the representative firm, $s(\theta, z^t) \in$

[27] We consider a tax scheme aiming at decentralizing the constrained efficient allocation. The arguments herein are similar to the ones used by Farhi et al. (2012).

$[0,1]$, at the price $q(z^t)$. His optimization problem is

$$\max_{c^y(\theta,z^t),c^o(\theta,z^{t+1}),e(\theta,z^t),s(\theta,z^t)} u\left(c^y(\theta,z^t)\right)+\beta\sum_{z^{t+1}\succ z^t}\pi(z^{t+1}|z^t)\beta u\left(c^o(\theta,z^t)\right)-\theta h(e(\theta,z^t))$$

s.t.

$$\begin{aligned} c^y(\theta,z^t)+q(z^t)s(\theta,z^t) &\leq z_t e(\theta,z^t)-T^y(z_t e(\theta,z^t)) \\ c^o(\theta,z^{t+1}) &\leq \left[0+T^o(z^{t+1})+q(z^{t+1})\right]s(\theta,z^t) \quad \forall z^{t+1}\succ z^t \end{aligned}$$

Note that the tax scheme $(T^y(z_t e(\theta,z^t)),T^o(z^{t+1}))_{z^t}$ is not allowed to be *directly* indexed on the agent's type, θ.

Proposition 8. *There exists a tax scheme that implements the second-best allocation.*

Workers want to buy a share of future production to consume when old. Normally, they would need only to buy shares and 'eat the dividends', as in the Lucas' (1978) model. The linear return on effort implies that the representative firm does not make any profits, and, of course, pays zero dividends. As there are no dividends, the government needs to promise transfers contingent on the firm's revenue and on the number of shares an agent holds. These transfers, financed by the labor tax, act as the fruits of a Lucas' tree. This implementation can be viewed as a *capitalization* social security system: agents buy a share $s_i(z^t)$ of the economy when young, and receive retirement payments that are directly proportional to the contribution they made when young.

Note that the return on investment $s_i(z^t)$,

$$r(z^{t+1})=\frac{T^o(z^{t+1})+q(z^{t+1})}{q(z^t)}$$

is contingent on the aggregate shocks. Interestingly, in many countries, pension funds are required to invest in 'safe' assets. If we take safe assets to mean risk-free assets, then taxes on their return must be state-contingent, otherwise, efficiency will not be attained.

6 Conclusion

Intergenerational risk-sharing and the role of social security in its implementation is not a novel policy concern – Enders and Lapan (1982); Ball and Mankiw (2007); Gottardi and Kubler (2011). What this paper adds to the literature is to address intergenerational risk-sharing using a mechanism design approach. This allows the relationship between incentive provision and risk-sharing which is at the heart of the problem independently of specific ad hoc assumptions on the type of policy instruments used.

The framework used in the paper is aimed not at realism but at isolating the crucial elements that must be taken into account for optimal risk-sharing assessment: a Mirrlees' (1971) environment with heterogeneous, privately observed, productivity shocks, and a publicly known aggregate shock.

From a policy perspective, a new development in social security design has been the adoption by several countries of Notional Defined Contribution systems. These systems bring to the table the possibility of easily adjusting pensions and contributions to economic and demographic shocks, thus making the characterization of optimal sharing rules all the more relevant. Yet, how good is an instrument if we do not have a framework to assess how best to use it? If we do not know how benefits and contributions should optimally be adjusted to aggregate shocks? We provide a conceptual framework for thinking about these questions.

In this paper growth has been a synonym of per capita growth: we have kept the size of the population fixed. In a Pay-as-you-go system with fixed contribution, the implicit rate of return on social security investment is the economy's growth rate; if we fix the social security contribution rate, then more growth is mechanically translated into higher per capita benefits.[28] The question we ask in this Section is whether this is optimal.

If we assume that the population increases over time in a deterministic fashion and that the share of agents of each type remains the same for all generations, i.e., $\forall t$, $f_t(\theta) = f(\theta)$. Then, it is possible to show that the main results remain valid and proceed with some comparative statics. Importantly, perfect risk-sharing means that pensions should vary with GDP per capita, not GDP.[29] This has consequences for implementation under a capitalization system.

References

Andrew Atkeson and Robert E. Jr Lucas. On efficient distribution with private information. *Review of Economic Studies*, 59:427–453, 1992. 39

Andrew Atkeson and Robert E. Jr Lucas. Efficiency and equality in a simple model of efficient unemployment insurance. *Journal of Economic Theory*, 66:64–68, 1995. 39

[28]More precisely, the rate of growth of aggregate labor earnings – see Murphy and Welch (1998).

[29]The proof follows the steps of our previous derivation and we omit for brevity.

Laurence Ball and Gregory N. Mankiw. Intergenerational risk sharing in the spirit of arrow, debreu and rawls with applications to social security design. *Journal of Political Economy*, 115(4):523 – 547, 2007. 2, 5, 8, 27

Nicholas Barr and Peter Diamond. *Pension Reform: A Short Guide*. Oxford University Press, 2009. 2

Spencer Bastani, Sören Blomquist, and Luca Micheletto. Public pensions in a multi-period mirrleesian income tax model. In Robert Holzmann and John Piggot, editors, *The Taxation of Pensions*. MIT Press, 2018. 15

Carlos da Costa and Vitor Farinha Luz. The private memory of aggregate uncertainty. *Review of Economic Dynamics*, 27:169–183, 2018. 6, 18

Gabrielle Demange. Sharing aggregate risks under moral hazard. *unpublished*, 2008. 6, 14, 15

Peter A Diamond. National debt in a neoclassical growth model. *American Economic Review*, 55:1125 – 1150, 1965. 5

Walter Enders and Harvey Lapan. Social security taxation and intergenerational risk sharing. *International Economic Review*, 23:647–658, 1982. 2, 5, 27

Emmanuel Farhi and Iván Werning. Insurance and taxation over the life cycle. *The Review of Economic Studies*, 80:596–635, 2013. 2

Emmanuel Farhi, Chirstopher Sleet, and Iván Werning. Nonlinear capital taxation without commitment. *Review of Economic Studies*, 79:1469–1493, 2012. 26

Mikhail Golosov, Narayana Kocherlakota, and Aleh Tsyvinski. Optimal indirect and capital taxation. *The Review of Economic Studies*, 70:569–587, 2003. 2

Roger H. Gordon and Hal R. Varian. Intergenerational risk sharing. *Journal of Public Economics*, 37:185–202, 1988. 2

Piero Gottardi and Felix Kubler. Social security and risk sharing. *Journal of Economic Theory*, 146:1078–1106, 2011. 5, 8, 27

Borys Grochulski and Narayana Kocherlakota. Nonseparable preferences and optimal social security systems. *Journal of Economic Theory*, 145:2055–2077, 2010. 2

Dirk Krueger and Felix Kubler. Intergenerational risk sharing via social security? *American Economic Review*, 96:737–755, 2006. 2

Robert E. Lucas. Asset prices in an exchange economy. *Econometrica*, pages 1429–1445, 1978. 27

James A. Mirrlees. An exploration in the theory of optimum income taxation. *Review of Economic Studies*, 38:175–208, 1971. 2, 5, 6, 22, 28

Kevin M. Murphy and Finis Welch. Perspectives on the social security crisis and proposed solutions. *American Economic Review*, 88(2):142–150, 1998. 28

Janusz A. Ordover and Edmund S. Phelps. The concept of optimal taxation in the overlapping-generations model of capital and wealth. *Journal of Public Economics*, 12:1 – 26, 1979. 2, 5

Christopher Phelan. Incentives and aggregate shocks. *The Review of Economic Studies*, 61:681–700, 1994. 5, 6

Florian Scheuer. Optimal asset taxes in financial markets with aggregate uncertainty. *Review of Economic Dynamics*, 16:405–420, 2013. 6

Marcelo Veracierto. Business cycle fluctuations in mirrlees economies: The case of i.i.d. shocks. Working Paper 2020-04, Federal Reserve Bank of Chicago, February 2020. 4, 6, 18, 24

Iván Werning. Optimal fiscal policy with redistribution. *Quartely Journal of Economics*, 122:925–967, 2007. 5, 6

A Proofs

A.1 Some Useful Definitions

Definition 6. Workers' ***consumption is separable*** at node z^t if, for all θ, there are functions $\bar{\eta}^y : z^t \longrightarrow \mathbb{R}_+$, $\phi^y : \Theta \times z^{t-1} \longrightarrow \mathbb{R}_+$ such that:

$$u'(c^y(\theta, z^t)) = \phi^y(\theta, z^{t-1})\bar{\eta}^y(z^t).$$

This definition means that z^{t-1} is a sufficient statistic for θ, under the density function $u'(c^y(\theta, z^t))$. An analogous definition holds for retirees and effort. We are interested in this property because consumption for either group, workers or retirees, is separable if and only if this group displays optimal risk-sharing.

Definition 7. An ***allocation is separable*** at z^t if workers' consumption, retirees' consumption, and effort are all separable with $\bar{\eta}^y(z^t) = \bar{\eta}^o(z^t) = \frac{\bar{\eta}_e(z^t)}{z_t}$.

Note that if the allocation is separable, then the allocation *between* and *within* workers and retirees will display perfect risk-sharing. If the allocation is separable for all z^t, we just say that it is separable.

A.2 Lemmata

Lemma 1. *The allocation is separable in period t if and only if workers' consumption is separable at t or workers' effort is separable at t.*

Proof. The "only if" is by definition. Let us prove the "if" part. Suppose that workers' consumption is separable, then by equation (26), $\mu(\theta, z^t)$ will not depend on z_t. This implies that effort will also be separable. Finally, if we use $\lambda(z^t)$ equivalences, we will find that $\overline{\eta}^y(z^t) = \overline{\eta}^o(z^t) = \frac{\overline{\eta}^e(z^t)}{z_t}$. Therefore, the entire allocation will be separable. The proof that workers' effort is separable is analogous. □

Lemma 2. *Let $\mu(q,\tau) = ((\phi^y(\theta), \phi^o(\theta), \phi^e(\theta))_\Theta)_{i=1}^N$ be the solution of $\mathbb{P}_1$. Then, $\exists(q,\tau) \in \mathbb{R}_+^2$, such that μ is also solution of $\mathbb{P}_0$.*

Proof. of Lemma 2 Consider the program

$$\max \int_\Theta \left\{u(\phi^y(\theta)) + \alpha\beta u(\phi^o(\theta)) - \theta h(\phi^e(\theta))\right\} f(\theta) d\theta,$$

subject to $\dot{U}(\theta) = -h(\phi^e(\theta))$ and $\int_\Theta \{\phi^e(\theta) - \phi^y(\theta) - q\phi^o(\theta)\} \geq \tau$.

We change variables to transform the program into a concave one. First, define the inverse function: $g(\phi^y, \phi^o, U, \theta) = h^{-1}\left(\frac{u(\phi^y + \alpha\beta u(\phi^o) - U}{\theta}\right)$. Omitting arguments for brevity, we can rewrite our problem with the following Lagrangian

$$\mathbb{L} = \int_\Theta \{Uf + \lambda(g - \phi^y - q\phi^o - \tau)f - \dot{\mu}U + \mu h(g)\} d\theta$$

where we used $\mu(\overline{\theta}) = \mu(\underline{\theta}) = 0$. It is straightforward to check that the first-order conditions are the same from program $\mathbb{P}_0$. Let $q := \alpha\delta$. Now, we need to choose τ in order to make the resource constraints match. Let $g(\tau) = (1-q)\int_\Theta \phi^o(\theta,\tau) f(\theta)$, were $(\phi^o(\theta,\tau))_\Theta$ is the consumption of retirees associated with $\mu(q,\tau)$ for $q = \alpha\delta$. We know that $\phi^o(\theta,\tau) \leq \phi^o(\underline{\theta},\tau) \quad \forall\theta$. Therefore, $g(\tau) \leq (1-q)\phi^o(\underline{\theta},\tau)$.

We claim that, $\forall\tau$, $\phi^o(\underline{\theta},\tau)$ is bounded above by a constant M. Let $v_{FB}(\tau)$ be the aggregate utility associated with the first-best allocation. Note that

$$\begin{aligned}\Phi u(\phi^y(\underline{\theta},\tau)) - \overline{\theta} h(\phi^e(\underline{\theta},\tau)) &\leq \Phi u(\phi^y(\overline{\theta},\tau)) - \overline{\theta} h(\phi^e(\overline{\theta},\tau)) \\ &\leq \int_\Theta U(\theta) f(\theta) d\theta \leq v_{FB}(\tau) \leq v_{FB}(0).\end{aligned}$$

Therefore,

$$\Phi u(\phi^y(\underline{\theta},\tau)) \leq \overline{\theta} h(\phi^e(\underline{\theta},\tau)) + v_{FB}(0), \quad \text{and} \quad \frac{u(\phi'^y(\underline{\theta},\tau))}{\underline{\theta} h(\phi'^e(\underline{\theta},\tau))} = 1.$$

Towards a contradiction assume that $\phi^o(\underline{\theta},\tau)$ is unbounded. Then, as $\phi^o(\underline{\theta},\tau)$ becomes arbitrarily large, from the second equation, effort must become arbitrarily small, thus violating the first inequality. Finally, as $g(0) > 0$, and $g(\tau)$ is bounded above by $(1-q)M$, we have that $g : [0, M] \to [0, M]$ satisfies all the conditions for the Brower fixed point theorem. This implies that $\exists \tau^* \in [0, M]$, such that $g(\tau^*) = \tau^*$. Substituting on the resource constraint, we find that

$$\begin{aligned}
\int_\Theta \{\phi^e(\theta) - \phi^y(\theta) - q\phi^o(\theta)\} &\geq \tau, \\
\int_\Theta \{\phi^e(\theta) - \phi^y(\theta) - q\phi^o(\theta)\} &\geq (1-q)\int_\Theta \phi^o(\theta) f(\theta) d\theta, \qquad \text{and} \\
\int_\Theta \{\phi^e(\theta) - \phi^y(\theta) - \phi^o(\theta)\} &\geq 0
\end{aligned}$$

□

Lemma 3. *If an efficient allocation is devoid of memory, then it displays perfect risk-sharing and there exist functions, $\phi : \Theta \to \mathbb{R}_+$, such that*

$$\begin{aligned}
u'(c^y(\theta, z^t)) &= \phi(\theta)\overline{\eta}(z_t) \\
u'(c^o(\theta, z^t)) &= (\delta/\beta)\phi(\theta)\overline{\eta}(z_t) \\
\Omega(z^t) &= (\beta/\delta)^{1/\sigma},
\end{aligned}$$

where $\overline{\eta}(z_t) \propto z_t^{\frac{-\sigma\gamma}{\sigma+\gamma-1}}$.

Proof. of Lemma 3 If the allocation is devoid of memory, then by Corollary 1.3, workers' consumption displays perfect risk-sharing. Then, by Proposition 3 the allocation displays perfect risk-sharing. Finally, if we take the partial derivative on θ on equations (26 - 27), we conclude the proof. □

A.3 Proofs

Proof. of Proposition 1 To preserve concavity when we introduce the incentive compatibility constraints, we adopt a standard procedure by now of changing variables. Define the inverse functions $g(h(e)) := e$ for all $e \in [0, \infty)$, and $\varphi(u(c)) = c$, and use $h(\theta)$ and $u^y(\theta)$, and $u^o(\theta)$ instead of $e(\theta)$, $c^y(\theta)$ and $c^o(\theta)$ as the relevant choice variables. Then, omitting arguments for brevity, we write the Lagrangian

$$\mathbb{L} = \int_\Theta \Bigg\{ \Big[\nu + \delta \mathbb{E}_z v(\tau', z')\Big]\chi + \Big[\lambda\big(zg(h) - \tau - \varphi(\nu + \theta h - \beta \mathbb{E}_z u^o(\tau', z'))\big) + \sum_{z'} \gamma(z')(\tau'(z') - \varphi(u^o(\tau', z')))\Big] f - \dot{\mu}\nu + \mu h \Bigg\} d\theta$$

Where we are imposing the boundary conditions $\mu(\overline{\theta}) = \mu(\underline{\theta}) = 0$. The first-order conditions, respectively on $(\nu, h, u^o(\tau', z'), \tau')$ and the envelope condition on τ are

$$\dot{\mu}(\theta) = \chi(\theta) - \frac{\lambda f(\theta)}{u'(c^y(\theta))} \tag{20}$$

$$\mu(\theta) = \lambda \theta f(\theta) \left[\frac{1}{u'(c^y(\theta))} - \frac{z}{\theta h'(e(\theta))}\right] \tag{21}$$

$$\frac{\lambda \beta \pi(z'|z)}{u'(c^y(\theta))} = \frac{\gamma(z')}{u'(c^o(\theta, \tau', z'))} \tag{22}$$

$$\delta \pi(z'|z) v_\tau(\tau', z') = -\gamma(z') \tag{23}$$

$$v_\tau(\tau, z) = -\lambda \tag{24}$$

Substituting (23-24) in (22), we find that

$$\frac{\beta u'(c^o(\theta, \tau, z, \tau', z'))}{u'(c^y(\theta, \tau, z))} = \delta \frac{\lambda(\tau', z')}{\lambda(\tau, z)} \tag{25}$$

Integrating (20) in θ and using $\mu(\underline{\theta}) = 0$, we obtain

$$\mu(\theta) = \int_{\underline{\theta}}^{\theta} \left[\frac{\chi(s)}{f(s)} - \frac{\lambda}{u'(c^y(s))}\right] f(s) ds \tag{26}$$

$$= \int_{\underline{\theta}}^{\theta} \left[\frac{\chi(s)}{f(s)} - \frac{\delta \lambda(z')}{\beta u'(c^o(s, z'))}\right] f(s) ds. \tag{27}$$

where in (27) we used (25). Finally, using that $\mu(\overline{\theta}) = 0$, we find that

$$\frac{1}{\mathbb{E}_z \left[1/u'(c^y(\theta, \tau, z))\right]} = \lambda(\tau, z) = \frac{\beta}{\delta} \frac{1}{\mathbb{E}_z \left[1/u'(c^o(\theta, \tau, z))\right]}$$

□

Proof. of Corollary 1.1-1.3 Immediate from Proposition 1. □

Proof. of Proposition 2 From Proposition 1 equation (10) we have

$$\mathbb{E}_t\left[c^y(\theta,z^t)^\sigma\right] = \frac{\delta}{\beta}\mathbb{E}_t\left[c^o(\theta,z^t)^\sigma\right].$$

All that one needs is now to divide the left-hand side by $\mathbb{E}_t\left[c^y(\theta,z^t)\right]$ and the right-hand side by $\mathbb{E}_t\left[c^o(\theta,z^t)\right]$ and rearrange to get (13).

□

Proof. of Corollary 2.1 Immediate from Proposition 2. □

Proof. of Proposition 3 Suppose that workers' consumption displays perfect risk-sharing, then it is separable, and by equation (26), $\mu(\theta,z^t)$ does not depend on z_t, which implies perfect risk-sharing between workers and retirees.

For the other direction, note that from the equality between (26) and (27)

$$\frac{\dot{\mu}(\theta,z^t)-f(\theta)}{\dot{\mu}(\theta,z^{t-1})-f(\theta)} = \frac{\beta}{\delta}\frac{u'(c^o(\theta,z^t))}{u'(c^y(\theta,z^t))}.$$

If there is perfect risk-sharing between workers and retirees, the right-hand side of (28) does not depend on z_t. This implies that $\dot{\mu}(\theta,z^t)$ does not depend on z_t, from which we conclude that workers' consumption is separable. Therefore, the allocation displays perfect risk-sharing among workers if and only if it displays perfect risk-sharing between workers and retirees. □

Proof. of Proposition 4 From Corollary 1.2, we know that the replacement ratio is the same for all agents: $c^o(\theta,z^{t+1})/c^y(\theta,z^t) = C^o(z^{t+1})/C^y(z^t), \forall\theta$. This allows us to show that for any constrained efficient allocation, $u(c^y(\theta,z^t))+\beta\mathbb{E}_t[\xi_{t+1}u(c^o(\theta,z^{t+1}))] = \Phi(z^t)u(c^y(\theta,z^t))$, thus proving Proposition 4. □

Proof. of Proposition 5 We first construct a Pareto improving reform. Then we prove that it entails lower marginal tax rates and a lower moment condition.

Pareto superior allocation: We show how a local reform Pareto improving reform can be designed following a perturbation on Φ, $d\Phi > 0$. We simplify notation by omitting the dependence of all variables on z^t and by assuming that $z_t = 1$. To focus on gains unrelated to the direct effects that arise due to better growth perspectives we consider a reform that increases every agent's flow utility at youth, $u(c^y) - \theta h(n)$, while holding fixed entitlements of this and previous generations.

Moreover, we work with the dual variables $u(\theta)$ and $h(\theta)$. That is, without loss, we start from an allocation that we can describe as $\{v_0(\theta)\}_\theta = \{u_0(\theta) - \theta h_0(\theta)\}_\theta$ and consider a small perturbation in Φ, $d\Phi > 0$.

Following the perturbation $d\Phi$ we consider a new allocation $\{u_1(\theta) - \theta h_1(\theta)\}_\theta$ defined by

$$u_1(\theta) = u_0(\theta) + \delta(\theta), \quad h_1(\theta) = h_0(\theta) + \eta(\theta)$$

for $\eta(\theta) \geq 0$. We choose $\delta(\theta)$ and $\eta(\theta)$ in such a way that $\delta(\theta) = \theta\eta(\theta)$. That is, $\forall\theta$, $u_1(\theta) - \theta h_1(\theta) = u_0(\theta) - \theta h_0(\theta)$, and

$$\dot{u}_1(\theta) - \frac{\theta}{\Phi + d\Phi}\dot{h}_1(\theta) = 0,$$

i.e., incentive compatibility is guaranteed. Since the original allocation was incentive compatible, it suffices to guarantee that

$$\dot{u}_1(\theta) - \frac{\theta}{\Phi + d\Phi}\dot{h}_1(\theta) = \dot{u}_0(\theta) - \frac{\theta}{\Phi}\dot{h}_0(\theta),$$

or

$$\dot{u}_0(\theta) + \dot{\delta}(\theta) - \frac{\theta}{\Phi + d\Phi}\left[\dot{h}_0(\theta) + \dot{\eta}(\theta)\right] = \dot{u}_0(\theta) - \frac{\theta}{\Phi}\dot{h}_0(\theta).$$

Noting that $\dot{\delta}(\theta) = \theta\dot{\eta}(\theta) + \eta(\theta)$, we have a differential equation for $\eta(z)$. If we define

$$a := \frac{\Phi + d\Phi}{\Phi + d\Phi - 1}, \quad b = \frac{d\Phi}{\Phi\left[\Phi + d\Phi - 1\right]}$$

then

$$\eta(\theta) = -b\int_{\underline{\theta}}^{\theta} \dot{h}(s)\left[\frac{\theta}{s}\right]^a ds,$$

solves this differential equation if we impose $\eta(\underline{\theta}) = 0$. The second order necessary condition for incentive compatibility implies $\dot{h}(\theta) < 0$ for all θ, which guarantees $\eta(\theta) \geq 0$ for all θ.

The cost of such reform is

$$\int \left[C'(u_0(\theta))\delta(\theta) - N'(h(\theta))\eta(\theta)\right] f(\theta)d\theta = \int \left[\theta C'(u_0(\theta)) - N'(h(\theta))\right] \eta(\theta) f(\theta)d\theta < 0.$$

The reform preserves utility and saves resources. These idle resources may then be returned to the agents in an incentive-compatible way by choosing $u_2(\theta) = u_1(\theta) + \kappa$ for all θ.

Lower MTR's, lower M_σ^y In the first step of the reform, for all agents initially facing a positive marginal tax rate we have increased both effort and consumption along their indifference curves, which means that marginal tax rates were reduced. In the second step, consumption has been increased without changes in effort, therefore further reducing marginal tax rates.

As for the moment condition, recall the total change in young age consumption is

$$\int C'(u(\theta))\Delta u(\theta) f(\theta)d\theta = \Delta C \tag{28}$$

Noting that $C'(u(\theta)) = C(u(\theta))^\sigma$, we have[30]

$$C^\sigma \int s(\theta)^\sigma \Delta u(\theta) f(\theta)d\theta = \Delta C. \tag{29}$$

It is also possible to re-write (28) as

$$(1-\sigma)\int C(u(\theta))\frac{\Delta u(\theta)}{u(\theta)} f(\theta)d\theta = \Delta C,$$

which gives us

$$\frac{\Delta C}{C} = (1-\sigma)\mathbb{E}\left[\frac{\Delta u(\theta)}{u(\theta)}\right] + (1-\sigma)\,\mathrm{Cov}\left(\frac{C(u(\theta))}{C}, \frac{\Delta u(\theta)}{u(\theta)}\right). \tag{30}$$

Next, let $\kappa := \int \Delta u(\theta) f(\theta)d\theta$, then, we can re-write (29) as

$$\Delta C = C^\sigma \mathbb{E}\left[s(\theta)^\sigma \Delta u(\theta)\right] = C^\sigma \left\{M_\sigma \kappa + \mathrm{Cov}\left(s(\theta)^\sigma, \Delta u(\theta)\right)\right\}$$

[30] $C(u) = [(1-\sigma)u]^{\frac{1}{1-\sigma}}$, and $C'(u) = [(1-\sigma)u]^{\frac{\sigma}{1-\sigma}}$

for $M_\sigma = \mathbb{E}\left[s(\theta)^\sigma\right]$, or $M_\sigma = C^{-\sigma}\int C(u(\theta))^\sigma f(\theta)d\theta$. In which case,

$$\begin{aligned}\Delta M_\sigma &= -\sigma C^{-\sigma-1}\int C(u(\theta))^\sigma f(\theta)d\theta\Delta C + \\ &\qquad \sigma C^{-\sigma}\int C(u(\theta))^{\sigma-1}C'(u(\theta))\Delta u(\theta)f(\theta)d\theta \\ &= -\sigma C^{-\sigma}\frac{\Delta C}{C}\int C(u(\theta))^\sigma f(\theta)d\theta + (1-\sigma)\sigma C^{-\sigma}\int C(u(\theta))^\sigma\frac{\Delta u(\theta)}{u(\theta)}f(\theta)d\theta.\end{aligned}$$

Using $s(\theta)^\sigma = c(\theta)^\sigma C^{-\sigma}$ and the definition of m_σ, we get

$$\begin{aligned}\Delta M_\sigma &= -\sigma\frac{\Delta C}{C}M_\sigma + (1-\sigma)\sigma C^{-\sigma}\int C(u(\theta))^\sigma\frac{\Delta u(\theta)}{u(\theta)}f(\theta)d\theta \\ &= -\sigma\frac{\Delta C}{C}M_\sigma + (1-\sigma)\sigma\int s(\theta)^\sigma\frac{\Delta u(\theta)}{u(\theta)}f(\theta)d\theta \\ &= -\sigma\frac{\Delta C}{C}M_\sigma + (1-\sigma)\sigma\mathbb{E}\left[s(\theta)^\sigma\frac{\Delta u(\theta)}{u(\theta)}\right].\end{aligned}$$

Finally,

$$\Delta M_\sigma = -\sigma\frac{\Delta C}{C}M_\sigma + (1-\sigma)\sigma M_\sigma\mathbb{E}\left[\frac{\Delta u(\theta)}{u(\theta)}\right] + (1-\sigma)\sigma\operatorname{Cov}\left(s(\theta)^\sigma, \frac{\Delta u(\theta)}{u(\theta)}\right).$$

Now, using (30), we get

$$\begin{aligned}\Delta M_\sigma = -\sigma\left\{(1-\sigma)\mathbb{E}\left[\frac{\Delta u(\theta)}{u(\theta)}\right] + (1-\sigma)\operatorname{Cov}\left(\frac{C(u(\theta))}{C}, \frac{\Delta u(\theta)}{u(\theta)}\right)\right\}M_\sigma + \\ (1-\sigma)\sigma M_\sigma\mathbb{E}\left[\frac{\Delta u(\theta)}{u(\theta)}\right] + (1-\sigma)\sigma\operatorname{Cov}\left(s(\theta)^\sigma, \frac{\Delta u(\theta)}{u(\theta)}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\Delta M_\sigma &= \sigma(1-\sigma)M_\sigma\left\{\operatorname{Cov}\left(\frac{s(\theta)^\sigma}{M_\sigma}, \frac{\Delta u(\theta)}{u(\theta)}\right) - \operatorname{Cov}\left(s(\theta), \frac{\Delta u(\theta)}{u(\theta)}\right)\right\} \\ &= \sigma(1-\sigma)M_\sigma\left\{\operatorname{Cov}\left(\frac{s(\theta)^\sigma}{M_\sigma} - s(\theta), \frac{\Delta u(\theta)}{u(\theta)}\right)\right\}\end{aligned}$$

If $\sigma > 1$ ($\sigma < 1$) then the term in brackets is positive (negative), which then implies $\Delta M_\sigma < 0$. To see this, assume for concreteness that $\sigma > 1$. If we define the distributions F^σ and F^s using s^σ/M_σ as their Radon-Nicodym derivatives with respect to F, then F^σ

is a mean preserving spread of F^s. Let θ_0 be the value at which these distributions cross. Then because $\Delta u(\theta)$ is increasing, $\Delta u(\theta)/u(\theta) < \Delta u(\theta_0)/u(\theta)$ for all $\theta < \theta_0$ and $\Delta u(\theta)/u(\theta) > \Delta u(\theta_0)/u(\theta)$ for all $\theta > \theta_0$. Finally note that $\Delta u(\theta_0)/u(\theta) < \Delta u(\theta_0)/u(\theta')$ for all $\theta < \theta_0$, $\theta' > \theta_0$.

□

Proof. of Proposition 6 By assumption the allocation displays perfect risk-sharing, therefore by Lemma 1, it is separable, $u'(c^y(\theta, z^t)) = \phi^y(\theta, z^{t-1})\overline{\eta}(z^t)$, $u'(c^o(\theta, z^t)) = \phi^o(\theta, z^{t-1})\overline{\eta}(z^t)$, and $h'(e(\theta, z^t)) = \phi^e(\theta, z^{t-1})z_t\overline{\eta}(z^t)$. Using the resource constraint,

$$\frac{\int_\Theta \left[u'^{-1}\left(\phi^y(\theta, z^{t-1})\overline{\eta}(z^t)\right) + u'^{-1}\left(\phi^o(\theta, z^{t-1})\overline{\eta}(z^t)\right)\right] f(\theta)d\theta}{\int_\Theta h'^{-1}\left(\phi^e(\theta, z^{t-1})z_t\overline{\eta}(z^t)\right) f(\theta)d\theta} = z_t \tag{31}$$

Define the function $m(z^t, \overline{\eta}(z^t))$ as the left-hand side of (31). $m(\cdot, \cdot)$ is decreasing in both arguments, so $\overline{\eta}(z^t)$ must be a decreasing function of z_t.

If preferences are iso-elastic, we will find that $\overline{\eta}(z^t) \propto z_t^{\frac{-\sigma\gamma}{\gamma+\sigma-1}}$, the same function found at the first best. □

Proof. of Corollary 6.1 Immediate from Proposition 6. □

Proof. of Corollary 6.2 Because the allocation displays perfect risk-sharing, it is separable. Therefore, from Proposition 6, $c^y(\theta, z^t) = \phi^y(\theta, z^{t-1})\eta(z_t)$ and $e(\theta, z^t) = \phi^e(\theta, z^{t-1})\frac{\eta(z_t)}{z_t}$, for $\eta(z_t) = z_t^{\frac{\gamma}{\sigma+\gamma-1}}$. Using (15), $v(\theta, z^t) = \left\{\Phi(z^t)u(\phi^y(\theta, z^{t-1})) - \theta h(\phi^e(\theta, z^{t-1}))\right\}\eta(z_t)^{1-\sigma}$.

The incentive constraints,

$$\Phi(z^t)u'(\phi^y(\theta, z^{t-1}))\dot{\phi^y}(\theta, z^{t-1}) = h'(\phi^e(\theta, z^{t-1}))\dot{\phi}(\theta, z^{t-1}),$$

therefore, imply that $\Phi(z^t)$ can not vary with z_t. □

Proof. of Proposition 7 Assume that the allocation displays perfect risk sharing, and define $\alpha(z_t) := \mathbb{E}_t\left[\eta\left(z_{t+1}/z_t\right)^{1-\sigma}\right]$. Lemma 3, then, implies

$$\begin{aligned} c^y(\theta, z^t) &= \phi(\theta)\eta(z_t) \\ c^o(\theta, z^t) &= \left(\frac{\beta}{\delta}\right)^{1/\sigma}\phi^y(\theta)\eta(z_t), \qquad \text{and} \\ e(\theta, z^t) &= \phi^e(\theta)\frac{\eta(z_t)}{z_t}, \end{aligned}$$

where $\eta(z_t) = z_t^{\frac{\gamma}{\sigma+\gamma-1}}$. Moreover,

$$\lambda(z^t) = \frac{\eta(z_t)^{-\sigma}}{\mathbb{E}[\phi^y(\theta)^\sigma]} = \tilde{\lambda}\eta(z_t)^{-\sigma}, \qquad \text{and} \qquad \mu(\theta, z^t) = \mu(\theta).$$

From (26), we can write $\phi^e(\theta)$ as a function of $(\phi^y(\theta))_\Theta$, using the resource and incentive constraints,

$$\begin{aligned} \Phi u'(\phi^y(\theta))\dot{\phi}^y(\theta) &= \theta h'(\phi^e(\theta))\dot{\phi}^e(\theta) \\ \left[1 + \left(\frac{\beta}{\delta}\right)^{1/\sigma}\right]\mathbb{E}(\phi^y(\theta)) &= \mathbb{E}[\phi^e(\theta)]. \end{aligned}$$

The question is whether there is a function $\phi(\theta)$ that satisfies this system. Note that, if $\Phi(z^t)$ varied with z^t, there would be no hope of finding such a function. Also, with iso-elastic preferences, the utility functions too would be separable.

Indeed, defining $\phi^o(\theta) := \left(\frac{\beta}{\delta}\right)^{1/\sigma}\phi(\theta)$, yields

$$\begin{aligned} v(\theta, z^t) &= \{u(\phi^y(\theta)) + \alpha\beta u(\phi^o(\theta)) - \theta h(\phi^e(\theta))\}\,\eta(z_t)^{1-\sigma} \\ v(\theta, z^t) &= \tilde{v}(\theta)\eta(z_t)^{1-\sigma} \end{aligned}$$

Moreover,

$$W(z^t) = \int_\Theta \tilde{v}(\theta)f(\theta)d\theta\eta(z_t)^{1-\sigma} \quad \text{and} \quad W(z^t) = \tilde{W}\eta(z_t)^{1-\sigma},$$

which using, $\mathbb{E}\left[\eta\left(z_{t+1}\right)^{1-\sigma}\right] = \alpha^{t+1}\eta\left(z_0\right)^{1-\sigma}$, leads to

$$\begin{aligned} \sum_t\sum_{z^t}\delta^t\pi(z^t)W(z^t) &= \tilde{W}\sum_t\delta^t\mathbb{E}\left[\eta(z_t)^{1-\sigma}\right] \\ \sum_t\sum_{z^t}\delta^t\pi(z^t)W(z^t) &= \tilde{W}\sum_t(\alpha\delta)^t\eta\left(z_0\right)^{1-\sigma} \end{aligned}$$

Expected utility is therefore bounded, if and only if, $\alpha\delta < 1$. From now on, we make this assumption. Also, for notation clarity, we take $z_0 = 1$.

In order to show that there exists $(\phi(\theta))_\Theta$ that solves this system, we will write an auxiliary problem that follows the spirit of the ones in (Atkeson and Lucas, 1992) and (Atkeson and Lucas, 1995). Assume that there is a sequence of planners, each one responsible to optimize a single generation welfare $\tilde{W}$, subject to incentive constraints and a modified resource constraint, that considers the intertemporal cost of providing utility to this generation. Formally, each planner is solving program $\mathbb{P}_1$, comprised of choosing $(\phi^y(\theta), \phi^o(\theta), \phi^e(\theta))_\Theta$, to maximize

$$\int_\Theta \{u(\phi^y(\theta)) + \alpha\beta u(\phi^o(\theta)) - \theta h(\phi^e(\theta))\} f(\theta)d\theta,$$

subject to $\dot{U}(\theta) = -h(\phi^e(\theta))$ and $\int_\Theta \{\phi^e(\theta) - \phi^y(\theta) - q\phi^o(\theta)\}$, where $\tau > 0$ and q capture the intertemporal cost that the planner of this generation is facing in order to provide utility when this generation becomes old.

We then use Lemma 2 to connect this problem with the one we are truly interested in and a standard result proves that this problem has a solution. Lemma 2 is formally proved below, but we want to give an intuition of their results first.

$\mathbb{P}_1$ is a concave program under the usual variable transformation. Therefore, again, we can apply the Kuhn-Tucker Theorem. In Lemma 2, we verify that taking $q = \alpha\beta$, the first-order conditions of $\mathbb{P}_1$ will generate the same incentive constraints as the original problem $\mathbb{P}_0$ we are truly interested in. Finally, we show that for this q, $\exists \tau > 0$, such that $\tau = (1-q)\int_\Theta \phi^o(\theta) f(\theta)d\theta$, which connects the resource constraint of this one-generation planner with the original one, leading to the last constraint. As the first order conditions are *necessary* and *sufficient* for μ to be optimal, it also solves $\mathbb{P}_0$.

Therefore, from Lemma 2, to prove that our non-linear system has a solution, all we need to show is that the program $\mathbb{P}_1$ admits an optimal. A standard result shows that, under the space of non-decreasing $\phi^e(\theta)$ in the weak topology, the problem has a solution. □

Proof. of Corollary 7.1 We can just apply Proposition 7. A sketch for an alternative, simpler proof is the following: normalize consumption from workers-retirees by z_t. Then, the shock z_t will only enter the problem additively on the social welfare function. Because the problem is concave, the (transformed) solution won't be indexed on z^t, implying that the original allocation will be *devoid of memory* and *separable*.

□

Proof. of Proposition 8 For this proof, we use the superscript $\tilde{a}$ for the second-best allocation.

Impose the following labor tax

$$T^y(z_t e_i(z^t)) = \begin{cases} z_t e(\theta, z^t), & \text{if } e(\theta, z^t) \notin (\tilde{e}(\theta, z^t))_\Theta \\ G(z_t e(\theta, z^t)), & \text{if } e(\theta, z^t) \in (\tilde{e}(\theta, z^t))_\Theta \end{cases},$$

where $G(z_t\tilde{e}(\theta, z^t)) < z_t\tilde{e}(\theta, z^t)$. This implies that the agent will choose $e(\theta, z^t) \in (\tilde{e}(\theta, z^t))_\Theta$.

Given the optimal effort, the agent chooses $(c^y(\theta, z^t), c^o(\theta, z^t), s(\theta, z^t))$ in a program that is concave and differentiable. The first-order conditions are, in this case,

$$\begin{aligned} u'(c^y(\theta, z^t)) &= \lambda(\theta, z^t), \\ \beta\pi(z^{t+1}|z^t)u'(c^o(\theta, z^{t+1})) &= \mu(\theta, z^{t+1}), \qquad \text{and} \\ q(z^t)\lambda(\theta, z^t) &= \sum_{z^{t+1}\succ z^t} \mu(\theta, z^{t+1})\left[T^o(z^{t+1}) + q(z^{t+1})\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} q(z^t)u'(c^y(\theta, z^t)) &= \mathbb{E}_t\left[\beta u'(c^o(\theta, z^{t+1}))\left(T^o(z^{t+1}) + q(z^{t+1})\right)\right], \\ q(z^t) &= \mathbb{E}_t\left[\frac{\beta u'(c^o(\theta, z^{t+1}))}{u'(c^y(\theta, z^t))}\left(T^o(z^{t+1}) + q(z^{t+1})\right)\right], \qquad \text{and} \\ q(z^t) &= \mathbb{E}_t\left[m(\theta, z^{t+1})\left(T^o(z^{t+1}) + q(z^{t+1})\right)\right], \end{aligned}$$

where $m(\theta, z^{t+1}) = \frac{\beta u'(c_i^o(z^{t+1}))}{u'(c_i^y(z^t))}$ is the stochastic discount factor.

We also known from (25) that $\tilde{m}(\theta, z^{t+1})$ does not depend on θ, so $\tilde{m}(\theta, z^{t+1}) = \tilde{m}(z^{t+1})$. From the resources constraints, we have that

$$\begin{aligned} \tilde{c}^o(\theta, z^{t+1}) &= \left[T^o(z^{t+1}) + q(z^{t+1})\right] s(\theta, z^t), \\ \int_\Theta \tilde{c}^o(z^{t+1})f(\theta)d\theta &= T^o(z^{t+1}) + q(z^{t+1}), \qquad \text{and} \\ \tilde{C}^o((z^{t+1}) &= T^o(z^{t+1}) + q(z^{t+1}). \end{aligned}$$

Hence, $q(z^t) = \mathbb{E}_t\left[\tilde{m}(z^{t+1})\tilde{C}^o(z^{t+1})\right]$. Now, the optimal government support per-share for retirees is given by $T^o(z^t) = \tilde{C}^o(z^t) - \mathbb{E}_t\left[\tilde{m}(z^{t+1})\tilde{C}^o((z^{t+1})\right]$, while the firm's share is given by $\tilde{s}(\theta, z^t) = \tilde{c}^o(\theta, z^t)/\tilde{C}^o(z^t)$. Finally,

$$G(z_t e(\theta, z^t)) = z_t e(\theta, z^t) - \tilde{c}^y(\theta, z^t) - \mathbb{E}_t\left[\tilde{m}(z^{t+1})\tilde{C}^o(z^{t+1})\right]\tilde{s}(\theta, z^t)$$

$\square$

B Numeric Implementation

We include the taste shock discussed in Section 4.1 in the description of our numeric procedure.

We want to solve:

$$v(\tau, x) = \max \int_\Theta \left[u(c^y(\theta)) - \theta h(e(\theta)) + \beta \mathbb{E}_x \xi' u(c^o(\theta, x'))\right] f(\theta) d\theta \\ + \beta \mathbb{E}_x \xi' v(\tau'(x'), x'),$$

subject to

$$\int_\Theta c^y(\theta) f(\theta) d\theta + \tau \leq z \int_\Theta e(\theta) f(\theta) d\theta,$$

$$\tau'(x') \geq \int_\Theta c^o(\theta, x') f(\theta) d\theta,$$

and

$$\dot{\nu}(\theta) = -h(e(\theta)).$$

Recall that

$$g(c_y, c_o, v, \theta, x) = h^{-1}\left(\frac{u(c^y) + \beta \mathbb{E}_x \xi' u(c^o(x')) - \nu}{\theta}\right) \quad \forall \quad e \in [0, \infty).$$

Then, omitting arguments for brevity, we write the Lagrangian

$$\mathbb{L} = \int_\Theta \Bigg\{ \Big[\nu + \beta \mathbb{E}_x \xi' v(\tau', x')\Big] f + \Big[\sum_{x'} \gamma(x')(\tau'(x') - c^o(x'))\Big] f \\ - \dot{\mu}\nu + \mu h(g) \Bigg\} d\theta + \lambda \left[z \int_\Theta g f d\theta - \int_\Theta c^y f d\theta - \tau\right]$$

The FOCs are (we omit (z, τ) for brevity):

$$f(\theta) - \dot{\mu}(\theta) - \frac{\mu(\theta)}{\theta} = \frac{z \lambda f(\theta)}{h'(e(\theta))\theta}, \qquad [v]$$

$$\frac{\mu(\theta)}{\theta}u'(c^y) = \lambda f(\theta)\left[1 - z\frac{u'(c^y)}{\theta h'(e(\theta))}\right], \qquad [c^y]$$

$$\frac{\mu(\theta)}{\theta}\beta\pi(x'|x)\xi' u'(c^o(x')) + \lambda f(\theta) z \frac{\beta\pi(x'|x)\xi' u'(c^o(x'))}{\theta h'(e(\theta))} = \gamma(x')f(\theta), \qquad [c^o(x')]$$

and

$$\beta\pi(x'|x)\xi' v_\tau(\tau', x') = -\gamma(x'). \qquad [\tau(x')]$$

The envelope is

$$v_\tau(\tau, x) + \lambda = 0$$

Reorganizing,

$$\begin{aligned}
\dot{\mu}(\theta) &= f(\theta)\left[1 - \frac{z\lambda}{h'(e(\theta))\theta}\right] - \frac{\mu(\theta)}{\theta} \\
c^y &= u'^{-1}\left(\frac{\lambda f(\theta)\theta h'(e(\theta))}{\mu(\theta)h'(e(\theta)) + z\lambda f(\theta)}\right) \\
c^o(z') &= u'^{-1}\left(\frac{-v_\tau(\tau', z')}{\lambda}u'(c^y)\right),
\end{aligned}$$

where the last equation follows from

$$\begin{aligned}
&-\gamma(x')f(\theta) + \frac{\mu(\theta)}{\theta}\beta\pi(x'|x)\xi' u'(c^o(x')) + \lambda f(\theta) z \frac{\beta\pi(x'|x)\xi' u'(c^o(x'))}{\theta h'(e(\theta))} = 0 \\
&\Rightarrow v_\tau(\tau', x')f(\theta) + \frac{\mu(\theta)}{\theta}u'(c^o(x')) + \lambda f(\theta) z \frac{u'(c^o(x'))}{\theta h'(e(\theta))} = 0 \\
&\Rightarrow -\frac{v_\tau(\tau', x')}{u'(c^o(x'))} = \frac{\mu(\theta)h'(e(\theta)) + \lambda f(\theta) z}{\theta f(\theta) h'(e(\theta))} \\
&\Rightarrow -\frac{v_\tau(\tau', x')}{u'(c^o(x'))} = \frac{\lambda}{u'(c^y)} \Rightarrow c^o(x') = u'^{-1}\left(\frac{-v_\tau(\tau', x')}{\lambda}u'(c^y)\right)
\end{aligned}$$

We are ready to construct our algorithm.

1. Guess a value function $v(\tau, x)$.

2. Guess $\lambda(\tau, x)$.

3. Guess $\nu(\underline{\theta}) = \underline{\nu}(\tau, x)$.

4. Guess (c^y, c^o). Use (c^y, c^o) to obtain a pair (e, ν) solving forward $\dot{\nu}(\theta) = -h(g)$, using $\nu(\underline{\theta}) = \underline{\nu}$. Then, solve backward the equation for $\dot{\mu}(\theta)$, using $\mu(\overline{\theta}) = 0$. Repeat this step until (c^y, c^o) converges.

5. Update $\tau'(x')$ and repeat step (4) until convergence.
6. Update $\underline{\nu}(\tau, x)$ and repeat step (4) until you get $\mu(\underline{\theta}) = 0$.
7. Update $\lambda(\tau, x)$ and repeat steps (4-5) until you get that Resource Constraints holds with equality.
8. Update $v(\tau, x)$ using the Bellman equation. Repeat steps (2-6) until convergence.

C Figures

Figure 1

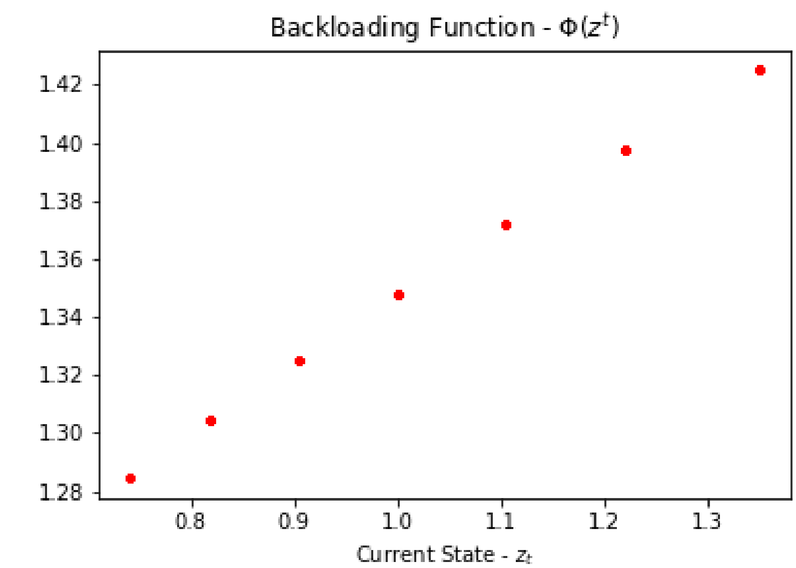


Figure 2

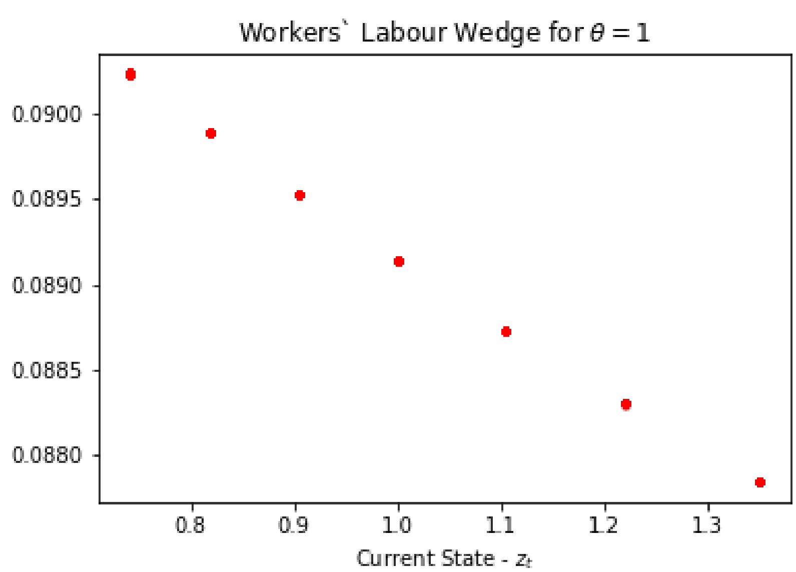


Figure 3

Figure 4

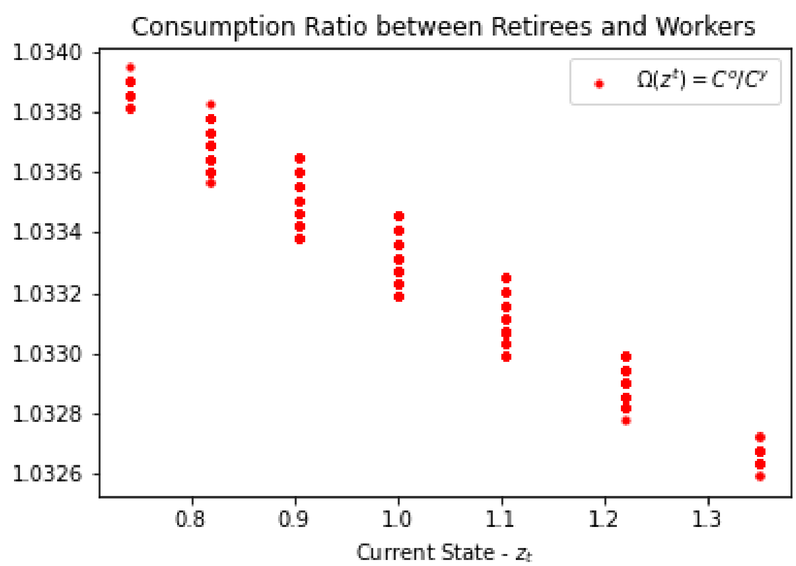


Figure 5

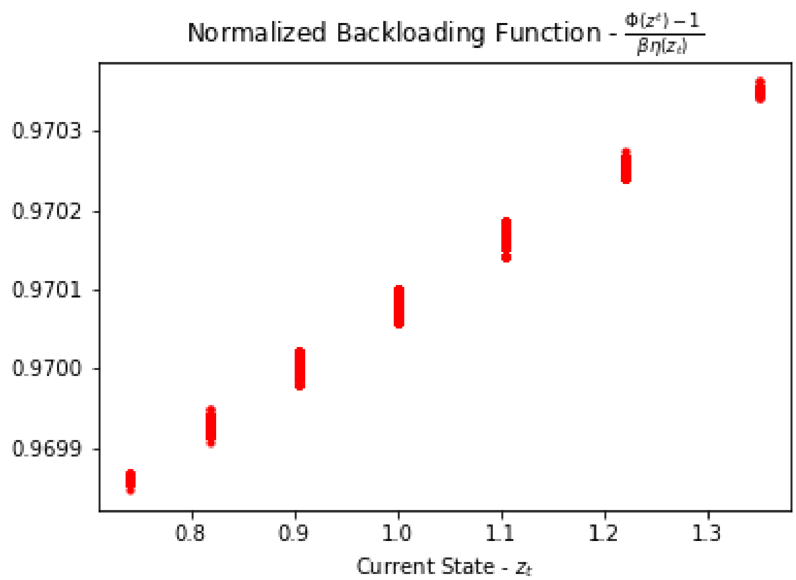


Figure 6

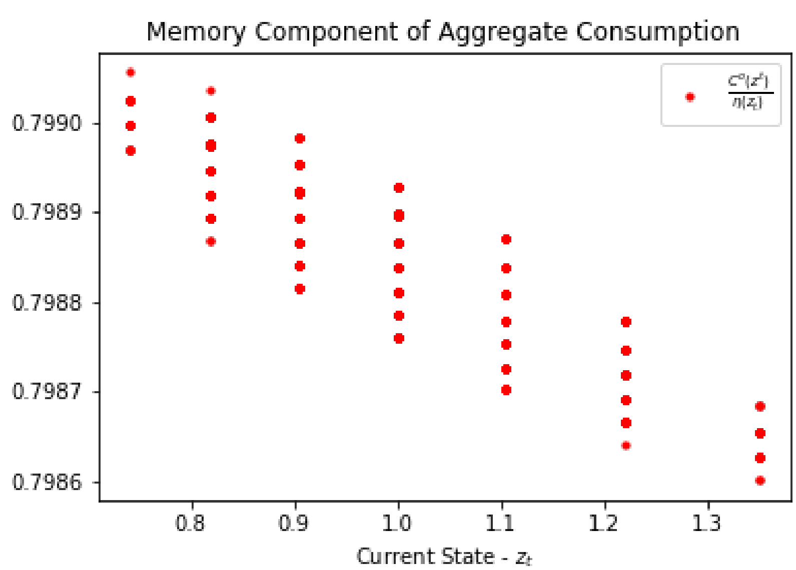


Figure 7

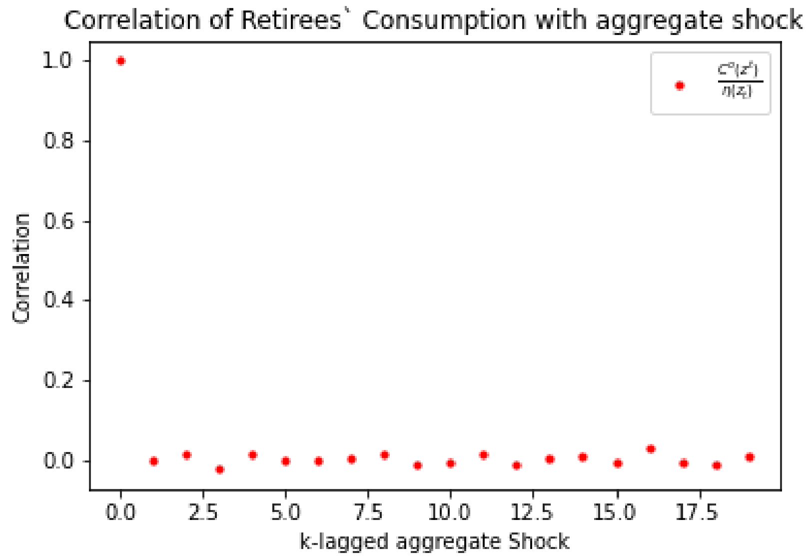
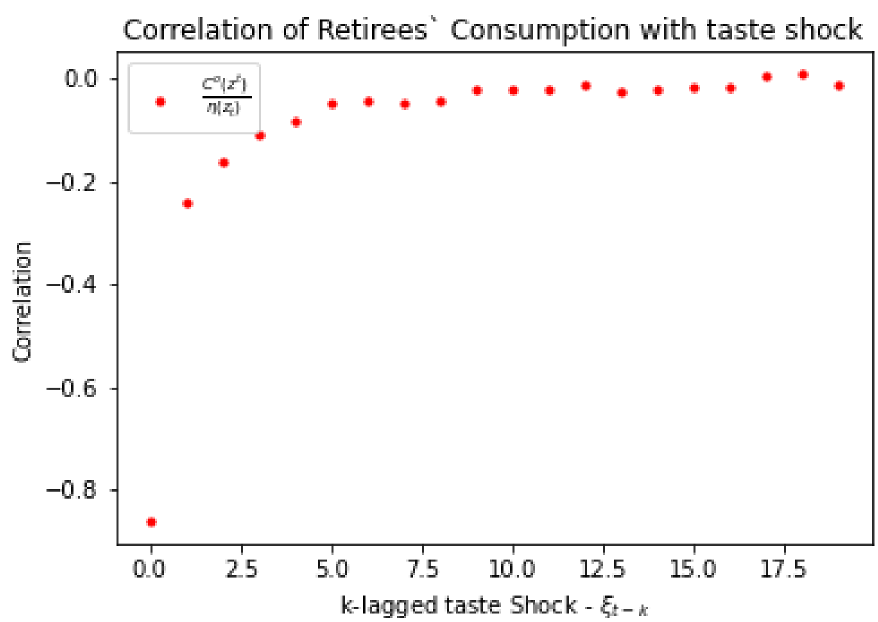


Figure 8



ONLINE APPENDIX

D Taste Shocks

We expand the model to consider taste aggregate taste shocks,

$$\mathscr{U}(c^y, c^o, e) = u(c^y) - \theta h(e) + \xi \beta u(c^o),$$

where ξ is a taste shock that hits every one of a given cohort.

Let ξ_t denote this common taste shock at period t, and $\xi^t = (\xi_1, ..., \xi_t)$, the history of taste shocks up to period t. Let $x_t := (z_t, \xi_t)$ and $x^t := (x_1, ..., x_t)$. We use $\pi(x^t)$ to denote the probability of history t, understanding that for every t there are a finite number of possible histories, x^t.

Consistent with the optimal risk-sharing idea, the planner solves a Utilitarian program,

$$\sum_t \sum_{z^t} \pi(z^t) \delta^t \Pi_{i=0}^t \xi_i \int_\Theta \nu(\theta, z^t) \chi(\theta) d\theta, \tag{32}$$

where we normalize $\xi_0 = 1$.

An ***allocation*** $(c^y(\theta, x^t), e(\theta, x^t), c^o(\theta, x^{t+1}))_{\theta \in \Theta, t, x^t}$ is ***feasible*** if for every period, t, and every history, x^t,

$$\int_\Theta \left\{ c^y(\theta, x^t) + c^o(\theta, x^t) \right\} f(\theta) d\theta \leq z_t \int_\Theta e(\theta, x^t) f(\theta) d\theta. \tag{33}$$

Let $(c^y(\theta, x^t), e(\theta, x^t), c^o(\theta, x^{t+1}))_{\theta \in \Theta}$ denote the allocation for cohort t if the economy experiences a history x^t. Then, the utility attained by a θ-agent who belongs to this cohort is

$$\nu(\theta, x^t) = u(c^y(\theta, x^t)) - \theta h(e(\theta, x^t)) + \beta \sum_{x^{t+1} \succ x^t} \pi(x^{t+1} | x^t) \xi_{t+1} u(c^o(\theta, x^{t+1})).$$

and

$$\nu(\hat{\theta} | \theta, x^t) = u(c^y(\hat{\theta}, x^t)) - \theta h(e(\hat{\theta}, x^t)) + \beta \sum_{x^{t+1} \succ x^t} \pi(x^{t+1} | x^t) \xi_{t+1} u(c^o(\hat{\theta}, x^{t+1})).$$

is the utility of an agent θ, at a given history x^t, that announces to be of type $\hat{\theta}$. Standard arguments imply that an allocation $(c^y(\theta, x^t), e(\theta, x^t), c^o(\theta, x^{t+1}))_{\theta\in\Theta,t,x^t}$ is ***incentive compatible*** if and only if $\forall\ \theta$, t, x^t, we have $\dot{\nu}(\theta, x^t) = -h(e(\theta, x^t))$ and $\dot{e}(\theta, x^t) \leq 0$.

E The Restricted Program

The planner's static program is

$$\max \int_Z \int_\Theta v(\theta, z) f(\theta)\pi(z) d\theta dz$$

s.t.

$$v(\theta, z) = \Psi(z)\frac{s(\theta)^{1-\sigma}}{1-\sigma} - \theta h(\theta, z),$$

$$\int s(\theta) f(\theta) d\theta = 1,$$

$$v(\theta, z) = v(\underline{\theta}, z) - \int_{\underline{\theta}}^{\theta} h(s, z) ds,$$

and

$$\left[\Psi(z) - \beta \int_Z \pi(\hat{z}|z) C^o(\hat{z})^{1-\sigma} d\hat{z}\right]^{\frac{1}{1-\sigma}} + \beta C^o(z) \leq z \int_\Theta N\left(h(\theta, z)\right) f(\theta) d\theta.$$

We start with the first order condition with respect to $\Psi(z)$,

$$\int_\Theta \alpha(\theta, z)\frac{s(\theta)^{1-\sigma}}{1-\sigma} d\theta = -\lambda(z)\frac{C^y(z)^\sigma}{1-\sigma}, \tag{34}$$

and with respect to $C^o(\hat{z})$,

$$-\beta \int_Z \lambda(z)\pi(\hat{z}|z) C^y(z)^\sigma C^o(\hat{z})^{-\sigma} dz + \beta\lambda(\hat{z}) = 0.$$

Let $\tilde{\lambda}(z) = \lambda(z)/\pi(z)$, then

$$-\beta \int_Z \tilde{\lambda}(z)\pi(\hat{z}|z)\pi(z) C^y(z)^\sigma C^o(\hat{z})^{-\sigma} dz + \beta\lambda(\hat{z}) = 0,$$

which, after noting that

$$-\beta\pi(\hat{z})\int_Z \tilde{\lambda}(z)C^y(z)^\sigma C^o(\hat{z})^{-\sigma}dz + \beta\lambda(\hat{z}) = 0,$$

can be written as

$$-\beta\int_Z \tilde{\lambda}(z)C^y(z)^\sigma C^o(\hat{z})^{-\sigma}dz + \beta\tilde{\lambda}(\hat{z}) = 0.$$

Now, the FOC with respect to $v(\theta, z)$, yields

$$\pi(z)f(\theta) + \alpha(\theta, z) - \mu_\theta(\theta, z) = 0,$$

and, with respect to $s(\theta)$,

$$-\alpha(\theta, z)\Psi(z)s(\theta)^{-\sigma} + \xi(z)f(\theta) = 0. \tag{35}$$

Multiplying (35) by $s(\theta)$ and integrating over θ

$$-\int_\Theta \alpha(\theta, z)\Psi(z)s(\theta)^{1-\sigma}d\theta + \xi(z)\int_\Theta f(\theta)s(\theta)d\theta = 0,$$

which, using (34) leads to

$$\lambda(z)C^y(z)^\sigma = \xi(z),$$

which gives us an expression for $\alpha(\theta, z)$

$$\alpha(\theta, z) = -\frac{\lambda(z)C^y(z)^\sigma}{\Psi(z)}s(\theta)^\sigma f(\theta). \tag{36}$$

Now, consider the first-order conditions with respect to $h(\theta, z)$,

$$\mu(\theta, z) = -\left[\alpha(\theta, z)\theta + \lambda(z)zN'(h(\theta, z))f(\theta)\right],$$

which, using (36) produces

$$\frac{\mu(\theta, z)}{\lambda(z)f(\theta)} = zN'(h(\theta, z))\left[\frac{C^y(z)^\sigma s(\theta)^\sigma\theta}{\Psi(z)zN'(h(\theta, z))} - 1\right]. \tag{37}$$

We need now only find expressions for $\mu(\theta, z)$ and $\lambda(z)$. Starting with the former,

$$\pi(z)F(\theta) + \int_{\underline{\theta}}^{\theta} \alpha(j,z)dj = \int_{\underline{\theta}}^{\theta} \mu_j(j,z)dj,$$

or

$$\int_{\underline{\theta}}^{\theta} \alpha(j,z)dj - F(\theta)\int_{\underline{\theta}}^{\bar{\theta}} \alpha(j,z)dj = \mu(\theta,z)$$

since

$$\pi(z) = -\int_{\underline{\theta}}^{\bar{\theta}} \alpha(j,z)dj,$$

as a consequence of $\mu(\bar{\theta}, z) = 0$. Then, using (36) we get

$$\frac{\lambda(z)C^y(z)^\sigma}{\Psi(z)}\left[F(\theta)\int_{\underline{\theta}}^{\bar{\theta}} s(j)^\sigma f(j)dj - \int_{\underline{\theta}}^{\theta} s(j)^\sigma f(j)dj\right] = \mu(\theta,z),$$

or

$$\frac{C^y(z)^\sigma}{\Psi(z)}F(\theta)\left\{\mathbb{E}\left[s(\theta)^\sigma\right] - \mathbb{E}\left[s(j)^\sigma | j \leq \theta\right]\right\} = \frac{\mu(\theta,z)}{\lambda(z)}.$$

Replacing the expression above in (37) yields

$$\left\{\mathbb{E}\left[s(j)^\sigma | j \leq \theta\right] - \mathbb{E}\left[s(\theta)^\sigma\right]\right\}\frac{C^y(z)^\sigma}{\Psi(z)f(\theta)}F(\theta) = zN'(h(\theta,z))\left[1 - \frac{C^y(z)^\sigma s(\theta)^\sigma \theta}{\Psi(z)zN'(h(\theta,z))}\right],$$

or

$$\left\{\mathbb{E}\left[s(j)^\sigma | j \leq \theta\right] - \mathbb{E}\left[s(\theta)^\sigma\right]\right\}\frac{F(\theta)}{f(\theta)\theta}s(\theta)^{-\sigma} = \frac{\Psi(z)zN'(h(\theta,z))}{C^y(z)^\sigma \theta s(\theta)^\sigma}\left[1 - \frac{C^y(z)^\sigma s(\theta)^\sigma \theta}{\Psi(z)zN'(h(\theta,z))}\right],$$

and finally,

$$\left\{\mathbb{E}\left[s(j)^\sigma | j \leq \theta\right] - \mathbb{E}\left[s(\theta)^\sigma\right]\right\}\frac{F(\theta)}{f(\theta)\theta}s(\theta)^{-\sigma} = \frac{\tilde{\tau}(\theta)}{1-\tilde{\tau}(\theta)}$$

where

$$\tilde{\tau}(\theta) = 1 - \frac{C^y(z)^\sigma s(\theta)^\sigma \theta}{\Psi(z)zN'(h(\theta,z))}$$

F Linear Effort

Assume that $\gamma = 1$, thus $h(e) = e$. We can rewrite (20-21) as

$$\dot{\mu}(\theta) = \chi(\theta) - \frac{\lambda f(\theta)}{u'(c^y(\theta))} \quad (38)$$

$$\mu(\theta) = \lambda\theta f(\theta)\left[\frac{1}{u'(c^y(\theta))} - \frac{z}{\theta}\right] \quad (39)$$

Substituting (38) in (39), we obtain

$$\theta\dot{\mu}(\theta) + \mu(\theta) = (\theta\dot{\mu(\theta)}) = \left[\frac{\chi(\theta)}{f(\theta)}\theta - \lambda z\right] f(\theta)$$

$$\Rightarrow \theta\mu(\theta) = \int_{\underline{\theta}}^{\theta}\left[\frac{\chi(s)}{f(s)}s - \lambda z\right] f(s)ds, \quad (40)$$

where in (40) we used $\mu(\underline{\theta}) = 0$.

Evaluating (40) at $\theta = \overline{\theta}$, we obtain

$$\lambda = \frac{\theta_\chi}{z}, \quad \text{for} \quad \theta_\chi \equiv \int_{\underline{\theta}}^{\overline{\theta}} \theta\chi(\theta)d\theta \quad (41)$$

Substituting (41) in (40), we obtain

$$\mu(\theta) = \frac{1}{\theta}\int_{\underline{\theta}}^{\theta}\left[\frac{\chi(s)}{f(s)}s - \theta_\chi\right] f(s)ds \quad (42)$$

Next, substituting (41) and (42) in (39), we obtain

$$\frac{\mu(\theta)}{\theta_\chi f(\theta)} = \frac{\theta}{zu'(c^y(\theta))} - 1,$$

which then implies

$$\frac{\theta}{zu'(c^y(\theta))} = \frac{\mu(\theta) + \theta_\chi f(\theta)}{\theta_\chi f(\theta)},$$

or

$$u'(c^y(\theta)) = \frac{\theta}{z}\frac{\theta_\chi f(\theta)}{\mu(\theta) + \theta_\chi f(\theta)}.$$

Hence,

$$c^y(\theta, z) \propto z^{1/\sigma}.$$

From (25)

$$\frac{\beta u'(c^o(\theta,\tau,z,\tau',z'))}{u'(c^y(\theta,\tau,z))} = \delta\frac{\lambda(\tau',z')}{\lambda(\tau,z)}$$

$$u'(c^o(\theta,z)) = \frac{\delta}{\beta}\frac{\theta}{z}\frac{\theta_\chi f(\theta)}{\mu(\theta)+\theta_\chi f(\theta)}$$

Thus, the SDF is given by

$$Q(z^t) \equiv \beta\frac{z_{t+1}}{z_t}$$

and

$$\Phi(z^t) = 1 + \beta\mathbb{E}_t\left[\left(\frac{\beta}{\delta}\right)^\sigma\left(\frac{z_t}{z_{t+1}}\right)^{\frac{\sigma-1}{\sigma}}\right].$$

Also, by incentive compatibility,

$$\Phi(z)u'(c^y(\theta,z))\dot{c}^y(\theta,z) = \theta\dot{e}(\theta,z) \Rightarrow$$

$$e(\theta,z) = e(\underline{\theta},z) + \Phi(z)\int_{\underline{\theta}}^{\theta} u'(c^y(s,z))\dot{c}^y(s,z)ds \Rightarrow$$

$$e(\theta,z) = e(\underline{\theta},z) + \Phi(z)z^{\frac{1-\sigma}{\sigma}}\kappa(\theta)$$

Substituting in the Resource Constraint, we get

$$\begin{aligned} z\int ef &= \alpha\int c^y f \\ ze(\underline{\theta},z) + \Phi(z)z^{\frac{1-\sigma}{\sigma}} &= \tilde{\alpha}z^{1/\sigma} \\ e(\underline{\theta},z) &= \tilde{\alpha}z^{\frac{1-\sigma}{\sigma}} - \Phi(z)\frac{z^{\frac{1-\sigma}{\sigma}}}{z} \end{aligned}$$