The role of international reserves in mitigating financial shocks

Aloisio Araujo[*] Gabriel Pestana[†] Vitor Costa[‡]

July 26, 2023

Abstract

In a traditional open macro model, we show that when we account for the common exchange rate puzzles, we also generate general equilibrium levels for foreign reserves and exchange rate volatility consistent with common emerging markets values. In such an environment, the country finds it optimal to issue debt in domestic currency to finance assets in foreign currency. In a model without such correction, we show that the optimal portfolio on foreign currency consists of a short position, *e.g.* a debt in foreign currency. This correction is done through financial friction and key calibration. We show this result using both a reduced form and a micro foundation for financial friction. To outline the intuition, all results are obtained with closed-form solutions.

Keywords: Exchange rate, International reserves, Financial friction

[*]EPGE/FGV - IMPA
[†]EPGE/FGV
[‡]IMPA

1 Introduction

The reasons behind countries' accumulation of reserves are still not fully understood and difficult to quantify. Some motives for holding foreign exchange reserves include instilling confidence in the national currency, mitigating disorderly market conditions, facilitating monetary policy implementation, building intergenerational assets, or influencing exchange rates. While there is a consensus on the cost of maintaining reserves in terms of the spread between domestic and foreign rates, there is limited agreement on the benefits, particularly regarding their quantitative extent. The complexity of quantifying external risks contributes to the challenge of assessing the role of reserves in maintaining risk premia.

This study contributes to the existing literature by demonstrating the usefulness of reserves as an asset, even in the absence of debt crises, binding constraints, or large shocks such as disasters. By enhancing a standard open macro model to replicate realistic exchange rate dynamics through the inclusion of a financial shock, foreign assets become desirable due to the resulting pricing structure in general equilibrium. Recent research by Oleg and Dmitri (2021) demonstrates that appropriately calibrating this shock can address several exchange rate puzzles encountered in conventional open macro models. Without the financial shock, the standard model fails to generate this pricing structure, leading the country to prefer issuing non-defaultable debt rather than acquiring international reserves. The underlying mechanism behind this feature lies in the standard productivity shock and/or financial market completeness, as they induce simultaneous exchange rate depreciation and increased consumption, which are typically negatively correlated in data. This positive correlation, known as the Backus-Smith Puzzle in exchange rate literature, is a common occurrence in open macro models. When the domestic endowment or productivity surpasses that of foreign countries, coupled with home bias, it becomes possible to accumulate more domestic consumption baskets relative to foreign consumption baskets. Due to home bias, these baskets differ in composition, with the former being predominantly focused on domestic goods and the latter on foreign goods. Consequently, a positive shock to domestic endowment leads to a higher relative price of foreign consumption baskets, indicating a real exchange rate depreciation. Simultaneously, domestic consumption increases. In such an environment, foreign bonds become assets that offer excess returns only in favorable states. As risk-averse agents prefer assets that generate excess returns in adverse states, they favor a short position in foreign bonds or non-defaultable debt denominated in foreign currency. While default optionality is abstracted for simplicity, this paper provides closed-form solutions to demonstrate these outcomes.

The inclusion of a persistent financial shock can modify or even reverse this behavior. A risk premium shock amplifies the return on foreign bonds, resulting in an increase in domestic real rates and an anticipated real exchange rate appreciation in the future to maintain uncovered interest rate parity, adjusted for the risk premium shock. This expected real exchange appreciation occurs through an initial significant depreciation that gradually reverses as the shock diminishes. Both movements align with a decline in domestic consumption, driven by higher present consumption prices compared to future prices and a displacement of domestic consumption by foreign demand, driven by the lower price of the domestic endowment. Consequently, the risk premium shock induces an inverse correlation between real exchange rates and consumption, and its integration into the model can resolve various exchange rate puzzles, as formally demonstrated by Oleg and Itskhoki (2021). This shock also prompts the preference for holding international reserves rather than debt in the domestic country for almost any positive level of financial shock volatility. As we refine the model to replicate more realistic exchange rate features, we observe an increase in reserve accumulation.

The risk premium shock represents a reduced form of financial market frictions within economies. One advantage of this reduced form is the presence of numerous microfoundations, with a significant portion of macro-finance literature striving to endogenize such frictions. [1]. Here's an improved version of the introduction segment:

In this paper, we present a typical microfoundation from the financial frictions literature that endogenizes the reduced form. Our model incorporates the noise trader and limits-to-arbitrage framework proposed by De Long, Shleifer, Summers, and Waldmann (1990), which has been adapted to the exchange rate market by Jeanne and Rose (2002). The model captures the essence of the idea that emerging countries are still developing their financial markets, characterized by a limited number of financial intermediaries and their risk aversion.

The microfoundation we provide consists of a positive mass of noisy traders who exogenously demand foreign bonds. This demand needs to be intermediated by a relatively small mass of risk-averse financial intermediaries, who require a risk premium for the transaction. We demonstrate that this microfoundation can generate the empirical feature that higher reserves lead to lower equilibrium volatility of exchange rates. The crucial parameters shaping this relationship are the mass of noisy traders and the mass and risk aversion of financial intermediaries. Although calibrating these parameters using data is challenging, they provide important insights into the maturity of financial markets, particularly in developing economies. By calibrating these parameters, we can obtain a general equilibrium that implies foreign reserves consistent with values observed in emerging markets and equilibrium exchange rate volatility half the size. Closed-form solutions are obtained to facilitate intuitive understanding of the results.

Traditionally, the literature addressing the reserves problem focuses on sovereign debt crises or sudden stops, treating international reserves as emergency savings for large crises.

[1] Examples include: exogenous preferences for foreign assets **Dekle, Jeong, and Kiyotaki (2014)**; Shock to the net worth of financial intermediaries **Hau and Rey 2006, Brunnermeier, Nagel, and Pedersen 2009, Gabaix and Maggiori 2015, Adrian, Etula, and Shin 2015**; Incomplete information, heterogeneous beliefs and expectational errors **Evans and Lyons (2002), Gourinchas and Tornell (2004) and Bacchetta and van Wincoop (2006)**

Alfaro and Kanczuk (2009) build a sovereign default model similar to Arellano (2008) to examine the joint accumulation of reserves and debt, but they do not obtain numerical results consistent with data. Subsequently, various papers emerge in a similar environment, aiming to improve numerical results by introducing additional assumptions such as debt maturity and default haircuts. For example, Bianchi, Hatchondo, and Martinez (2018) work with a long-term debt instead of a one-period debt, making defaultable debt and reserves less similar assets and actively reducing rollover risk. Sabbadini (2019) designs a similar model with risk-averse lenders and debt haircuts. These extensions can be calibrated to reproduce observable values of international reserves in emerging markets. More recently, Alfaro and Kanczuk (2018) extend their previous model by introducing a non-tradeable sector to capture exchange rate behavior, finding a calibration that improves their numerical results, although exchange rate modeling presents challenges due to distortions compared to real-world data.

Taking a different approach, Hur and Kondo (2016) model short-term international debt as a contract with international investors, where reserves are used to finance long-term investment. This model structure resembles Diamond and Dybvig (1983), wherein a liquidity shock leads to a sudden decision by international investors to stop rolling over debt. In this context, it becomes optimal for countries to save a portion of the resources raised through debt as reserves to prepare for future liquidity runs before investment maturity. Bianchi (2011) views sudden stops as a binding constraint on debt accumulation, introducing endogenous sudden stops through non-linear policy functions near the constraints. These constraints become binding when debt is high and endowment is low, resulting in a drastic reduction in debt issuance, referred to as a sudden stop. Arce, Bengui, and Bianchi (2022) extend this model by incorporating reserve accumulations as a macroprudential policy and introducing a financial shock that affects the binding constraint for debt.

The existing body of literature shares a common emphasis on the role of international reserves in facilitating consumption smoothing during periods of significant crises. Undoubtedly, this recognition of reserves as a valuable incentive aligns with the sensitivity and perpetual risk faced by financial markets. However, our research extends beyond this prevailing understanding by introducing an additional dimension to the significance of reserves—hedging. By complementing the existing literature, we shed light on reserves as a hedging mechanism, emphasizing their appeal even in the absence of large-scale crises. This perspective stems from the fact that reserves are denominated in foreign currency, prompting optimal debt issuance in domestic currency to finance foreign currency-denominated assets, driven by the hedging incentive provided by exchange rates. To ensure a comprehensive analysis, we also address the challenge of incorporating realistic exchange rate dynamics, drawing upon recent advancements that rectify the exchange rate puzzles often observed in traditional open macro models. It is important to note that in order to achieve meaningful insights, the exchange rate dynamics in our model must adhere to a level of minimal realism. Notably, this presents a challenge as the literature extensively documents the exchange rate puzzles commonly observed in traditional open macro models. To address this, we draw upon a recent paper by **Oleg and Itskhoki (2021) ?** that presents simple adjustments to rectify the main exchange rate puzzles in traditional open macro models, providing a robust foundation for our analysis.

2 Baseline Model

In this section, we introduce the baseline model, which serves as the foundation for our analysis. The baseline model represents a traditional open macro model characterized by the absence of frictions and only endowment fluctuations within an open economy. To incorporate financial friction into this baseline model, we introduce an exogenous shock in the return of foreign bonds.

The economy comprises two countries, each represented by a representative agent. Within the economy, there are two goods: good H and good F. We use subscripts to indicate the origin of the demanded good/bond and superscripts with a star to denote foreign household demand for these goods/bonds[2]. Both households combine these goods into baskets and derive utility from consuming these baskets. The domestic consumption (C_t) and foreign consumption (C_t^*) are given by equations 1 and 2, respectively:

Equation (1):

$$C_t = \left[(1-\gamma)^{\frac{1}{\theta}} C_{H,t}^{\frac{\theta-1}{\theta}} + \gamma^{\frac{1}{\theta}} C_{F,t}^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \tag{1}$$

Equation (2):

$$C_t^* = \left[(1-\gamma)^{\frac{1}{\theta}} C_{F,t}^{*\,\frac{\theta-1}{\theta}} + \gamma^{\frac{1}{\theta}} C_{H,t}^{*\,\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \tag{2}$$

In these equations, $\theta > 1$ represents the elasticity of substitution between goods H and F within each basket. A higher value of θ indicates that households can easily substitute one good for another in response to changes in relative prices, resulting in lower levels of terms of trade and real exchange rate volatility. Typically, empirical calibrations and estimations of open macro models find values around $\theta \approx 1.5$, which is considerably lower than the elasticity of substitution in New Keynesian models (around $9 \sim 11$).

Another crucial parameter is γ, which measures the degree of home bias. Lower values of γ indicate a larger presence of good H in the domestic consumption bundle compared to good F. Consequently, when the quantity of H goods in the economy exceeds that of F goods, households can form more C_t baskets relative to C_t^* baskets. In general equilibrium, the price of the C_t basket will be lower than that of the C_t^* basket, representing a real exchange rate depreciation. Similarly, when $\gamma \to 1/2$, the relative price of the C_t basket will closely resemble that of the C_t^* basket, irrespective of the quantities of home and foreign goods. Hence, lower values of γ are associated with higher volatility of the real exchange rate.

[2]For instance, $C_{H,t}$ represents the domestic household's demand for the good H, while $C_{H,t}^*$ represents the foreign household's demand for the same good.

From the cost minimization problem of a typical household allocating H and F goods in a basket, given prices, we have:

$$C_{H,t} = (1-\gamma)P_{H,t}^{-\theta}C_t, \quad C_{F,t} = \gamma P_{F,t}^{-\theta}C_t \tag{3}$$

$$C_{F,t}^{*} = (1-\gamma)P_{F,t}^{*-\theta}C_t^{*}, \quad C_{H,t}^{*} = \gamma P_{H,t}^{*-\theta}C_t^{*} \tag{4}$$

Where $P_{H,t}$, $P_{H,t}^{*}$ is the price of good H in domestic and foreign currency, respectively. Similarly for $P_{F,t}, P_{F,t}^{*}$. For simplicity, we assume that monetary policy fixes the nominal price levels, which are the prices of the domestic basket in local currency $P_t = 1$, and the foreign basket in foreign currency, $P_t^{*} = 1$. Therefore, by definition, real exchange rate and nominal exchange rate are equal[3]. From the price index definition, such that total expenditure for goods equals total bundle quantity:

$$P_t = 1 = \left[(1-\gamma)P_{H,t}^{1-\theta} + \gamma P_{F,t}^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{5}$$

$$P_t^{*} = 1 = \left[(1-\gamma){P_{F,t}^{*}}^{1-\theta} + \gamma {P_{H,t}^{*}}^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{6}$$

Terms of Trade and Real Exchange Rate

The baseline model allows the law of one price for each good. The law of one price implies that[4] $P_{H,t} = Q_t P_{H,t}^{*}$ and $P_{F,t} = Q_t P_{F,t}^{*}$. Let Q_t be the real exchange rate, which represents how many domestic consumption baskets C_t can be exchanged for one foreign consumption basket, C_t^{*}. This means that when Q_t increases, there is a real depreciation of the domestic

[3]Let Q_t and $\mathcal{E}_t$ be the real and nominal exchange rate, respectively. From the definition of real exchange rate $Q_t = P_t^{*}\mathcal{E}_t/P_t = \mathcal{E}_t$

[4]It may seem quite obvious, but actually this contributes to a Purchase Power Parity puzzle. In this specification, terms of trade are more volatile than the real exchange rates. There are alternatives such as local currency pricing, or pricing to market, that make the price faced by foreign households for the good H actually different from the price faced by domestic households, corrected by the nominal exchange rate.

currency. Terms of trade are defined by:

$$S_t = \frac{P_{F,t}}{P_{H,t}} \tag{7}$$

Terms of trade (or relative price) are defined such that an increase in S_t means that the domestic country has a consumption increase relative to the foreign country. This is because if S_t increases, then $P_{F,t}$ is higher than $P_{H,t}$, which, in this endowment economy, means that there are more goods H than F available in the economy, bringing the relative price of $P_{H,t}$ down. In such an environment, more C_t can be built over C_t^*, and the real exchange rate will be depreciated. To simplify model equations, define the following functions of the terms of trade:

$$\frac{1}{P_{H,t}} = \left[(1-\gamma) + \gamma S_t^{1-\theta}\right]^{\frac{1}{1-\theta}} \equiv g(S_t) \tag{8}$$

$$\frac{1}{P_{F,t}} = \left[(1-\gamma) S_t^{-(1-\theta)} + \gamma\right]^{\frac{1}{1-\theta}} \equiv h(S_t) \tag{9}$$

$$\frac{1}{P_{H,t}^*} = \left[\gamma + (1-\gamma) S_t^{1-\theta}\right]^{\frac{1}{1-\theta}} \equiv g^*(S_t) \tag{10}$$

$$\frac{1}{P_{F,t}^*} = \left[\gamma S_t^{-(1-\theta)} + (1-\gamma)\right]^{\frac{1}{1-\theta}} \equiv h^*(S_t) \tag{11}$$

To obtain these functions just divide the price index definitions by each individual price, and use the normalization of price indexes. Using the law of one price, we have that $Q_t = g^*(S_t)/g(S_t)$. Inverting this function we have the non-linear relation between terms of trade and real exchange rate:

$$S_t = \left[\frac{Q_t^{1-\theta}(1-\gamma) - \gamma}{1 - \gamma(1 + Q_t^{1-\theta})}\right]^{\frac{1}{1-\theta}} \tag{12}$$

Observe that if $\gamma = 1/2$, then $Q_t = g^*(S_t)/g(S_t) = 1, \forall t$. In such case, from , it follows[5] that $S_t = 1, \forall t$. The source of real exchange rate and relative price fluctuation in the model

[5]To see that apply L'Hopital rule for $\gamma \to 1/2$, given that $S_t(1/2)$ gives an indetermination.

is the home bias[6] $\gamma < 1/2$.

Market-Clear and Budget Constraints

Market clear occurs in good markets and bond markets. Every period domestic economy receives a stochastic endowment of Y_t, denominated in H goods. Similarly, the foreign economy receives Y_t^* denominated in F goods. Market-clear implies that aggregate demand toward each good must be equal to its aggregate supply:

$$Y_t = C_{H,t} + C_{H,t}^*, \quad Y_t^* = C_{F,t} + C_{F,t}^* \tag{13}$$

Using equations 3 and 4 we can write this conditions as function of S_t, C_t and C_t^*. We could rule out S_t using equation 7 and only work with real exchange rates Q_t, but we carry S_t to avoid larger expressions:

$$Y_t = (1-\gamma)g(S_t)^\theta C_t + \gamma g^*(S_t)C_t^* \tag{14}$$

$$Y_t^* = (1-\gamma)h^*(S_t)^\theta C_t^* + \gamma h(S_t)C_t \tag{15}$$

Let $B_{H,t}$ denote the domestic demand for home bonds, and $B_{F,t}$ represent the domestic demand for foreign bonds. Both bonds are denominated in domestic consumption units, C_t. Domestic bonds provide a return in domestic currency units in the next period, while foreign bonds yield a return in foreign currency units in the subsequent period. To simplify the analysis, we assume a constant price level for each economy $P_t = P_t^* = 1$, whereby bond payments are equivalent to the respective consumption baskets themselves.

[6] Clearly $\gamma > 1/2$ would also provide real exchange rate fluctuation. But $\gamma > 1/2$ implies more imported goods than domestic goods in the aggregate domestic consumption, a not consistent behavior with data. An additional force of real exchange rate fluctuation is pricing to market behavior, which is later included for numerical enhancement.

We measure the size of all portfolios in units of C_t. Thus, the real return of a domestic bond purchased in period $t-1$ is denoted as R_{t-1}, with its price at $t-1$ normalized to one. Similarly, the real return of a foreign bond purchased in period $t-1$ consists of the real return in the foreign currency, denoted as R^*_{t-1}, and the currency price variation during the period, represented by Q_t/Q_{t-1}. Assets quantity normalization is done by expressing both budget constraints in terms of the domestic basket C_t. This approach enables us to compare portfolios without the need to adjust for exchange rates.

In summary, the real returns are endogenously determined in equilibrium, but each bond comprises a risk-free asset. One bond pays in units of C_t, while the other pays in units of C^*_t. The stochastic component arises from the fact that a domestic household can only derive utility from consuming C_t units. Therefore, when the household receives C^*_t payments from a foreign bond, they must exchange it for C_t units in the market. The exchange rate, which exhibits stochastic behavior, plays a crucial role in determining the value of currency conversions. The domestic household budget constraint is expressed as follows:

$$C_t + B_{H,t} + B_{F,t} = \frac{P_{H,t}}{P_t} Y_t + R_{t-1} B_{H,t-1} + e^{\psi_{t-1}} \frac{Q_t}{Q_{t-1}} R^*_{t-1} B_{F,t-1} \tag{16}$$

Here, the term ψ_t represents an exogenous risk-premium shock, which serves as a reduced form of financial friction. The shock leads to an increase in the risk premium, resulting in a lower expected return on the exchange rate. In equilibrium, this is achieved through a significant current depreciation, with the expectation of slower appreciation in the future. The risk-premium shock affects real exchange rates, real interest rates, and subsequently domestic consumption. Similarly, foreign households face a risk-premium risk in allocating domestic bonds, resulting in all agents experiencing the same excess return, in a first-order approximation. The budget constraint for foreign households, denominated in units of domestic baskets C_t, can be expressed as:

$$Q_t C_t^* + B_{H,t}^* + B_{F,t}^* = Q_t \frac{P_{F,t}^*}{P_t^*} Y_t + e^{-\psi_{t-1}} R_{t-1} B_{H,t-1}^* + \frac{Q_t}{Q_{t-1}} R_{t-1}^* B_{F,t-1}^*$$

To simplify the analysis, we can utilize Walras' law to omit this equation and combine the household budget constraint 16 with the market-clearing condition for good H 13 and the property of the price index $C_t = P_{H,t} C_{H,t} + P_{F,t} C_{F,t}$ to rewrite the budget constraint as:

$$B_{H,t} + B_{F,t} = R_{t-1} B_{H,t-1} + e^{\psi_{t-1}} \frac{Q_t}{Q_{t-1}} R_{t-1}^* B_{F,t-1} + \left[P_{H,t} C_{H,t}^* - P_{F,t} C_{F,t} \right]$$

In this formulation, the last term on the right-hand side represents the current account, which is equivalent to exports minus imports. The financial account represents the liquid returns from previous bond positions. By rewriting the current account in terms of C_t, C_t^*, S_t, and Q_t, we can formulate equilibrium conditions with fewer variables:

$$B_{H,t} + B_{F,t} = R_{t-1} B_{H,t-1} + e^{\psi_{t-1}} \frac{Q_t}{Q_{t-1}} R_{t-1}^* B_{F,t-1} + \gamma \left[Q_t g^* (S_t)^{\theta-1} C_t^* - h(S_t)^{\theta-1} C_t \right] \quad (17)$$

In this equation, we substitute $C_{F,t}, C_{H,t}^*$ using equations 3 and 4, applied the law of one price, and then used equations 9, 10 to write in terms of the terms of trade. We assume a zero-net supply for both bonds. Bonds market clear are:

$$B_{H,t} + B_{H,t}^* = 0 \quad (18)$$

$$B_{F,t} + B_{F,t}^* = 0 \quad (19)$$

By Walras Law, we can omit the budget constraint for foreign households. If the domestic budget is satisfied, and the market clears of bonds and goods markets are satisfied, the foreign budget constraint must be satisfied from an excess demand condition, and prices will be such that supports this allocation.

Households

As previously mentioned, each country is populated by a representative infinitely lived household[7]. Domestic household maximizes the expected discounted instantaneous utility:

$$\max_{\{C_t, B_{H,t}, B_{F,t}\}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}, \quad \text{such that}$$

$$C_t + B_{H,t} + B_{F,t} = P_{H,t} Y_t + R_{t-1} B_{H,t-1} + e^{\psi_{t-1}} \frac{Q_t}{Q_{t-1}} R^*_{t-1} B_{F,t-1}$$

Foreign households face a similar problem, but we write their budget constraint in terms of the domestic consumption bundle, C_t. They maximize:

$$\max_{\{C^*_t, B^*_{H,t}, B^*_{F,t}\}} E_0 \sum_{j=0}^{\infty} \beta^t \frac{C_t^{*1-\sigma}}{1-\sigma} \quad \text{subject to}$$

$$Q_t C^*_t + B^*_{H,t} + B^*_{F,t} = Q_t P^*_{F,t} Y_t + e^{-\psi_t} R_{t-1} B^*_{H,t-1} + \frac{Q_t}{Q_{t-1}} R^*_{t-1} B^*_{F,t-1}$$

First-order conditions are conventional Euler equations for both households:

$$E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} R_t \right] = 1 \tag{20}$$

$$E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} R^*_t \frac{Q_{t+1}}{Q_t} e^{\psi_t} \right] = 1 \tag{21}$$

$$E_t \left[\beta \left(\frac{C^*_{t+1}}{C^*_t} \right)^{-\sigma} R^*_t \right] = 1 \tag{22}$$

[7] This is just a reduced form of saying that the world is populated by a continuum of households and a continuum of countries, with the notion of the small open economy where foreign countries behave similarly. For a more detailed exposition of such mapping between economies, see Paolo Cavallino (AEJ:Macroeconomics, 2019)

$$E_t\left[\beta\left(\frac{C^*_{t+1}}{C^*_t}\right)^{-\sigma}\left(\frac{Q_t}{Q_{t+1}}\right)R_t e^{-\psi_t}\right]=1 \tag{23}$$

2.1 Equilibrium

Definition 1 *An equilibrium are functions* $\{C_t, C_t^*, Q_t, S_t, B_{H,t}, B^*_{H,t}, B_{F,t}, B^*_{F,t}, R_t, R^*_t\}$ *defined over the states* $(Y_t, Y_t^*, \psi_t, B_{H,t-1}, B_{F,t-1})$ *such that satisfies equations 2, 14, 15, 17, 18, 19, 20-23. Those equations correspond to 1 terms of trade equation, 2 goods market-clear, 2 bonds market-clear, 4 Euler equations, and 1 budget constraint.*

To provide intuition on the role of the ψ_t shock through closed-form solutions, we will work with a first-order approximation of such equilibrium. To identify portfolios in this approximation, we use the technique of **DEVEREUX and SUTHERLAND (2014)**. They show how to obtain the *zero-order* term of a Taylor series expansion of portfolios policy function. Therefore, in the first-order approximated solution, this value will be the ergotic mean of portfolios. If the first-order approximation is good enough, the *zero-order* portfolio should be closer to the true ergotic mean of portfolio ergotic distribution.

It is a well-known fact that incomplete markets open economy are not stationary, at least in the first-order accurate solution[8]. Therefore, there is no point in obtaining a steady-state portfolio because the economy will not be close to that point in any simulation. However, in the appendix, we adjust preferences by a Uzawa-Epstein factor to induce stationarity and obtain portfolios very similar to the ones obtained here. This stationarity is anchored by a parameter η, and portfolios preserve a continuity property when $\eta \to 0$. Therefore, all the intuition we obtain here is still valid in the stationary world.

To obtain the steady state we must analyze a version of the system that is neither dynamic nor stochastic. In this system, asset positions are not identifiable. Any position is actually

[8]See *Grohé and Uribe JIE (2003)*

a consistent solution. Using **DEVEREUX and SUTHERLAND (2014)** method, we find such a point that is consistent with first-order consumption and returns dynamics and a second-order risk aversion arising from preferences. Their method consists of noticing that, in a second-order approximation of Euler equations[9], we get an equation that depends on a combination of only first-order terms:

$$E_t \left(\sigma c^R_{t+1} - q_{t+1}\right) r^X_{t+1} = 0 \tag{24}$$

Where $c^R_{t+1} \equiv c_{t+1} - c^*_{t+1}$, $r^X_{t+1} \equiv q_{t+1} - q_t + r^*_t - r_t + \psi_t$, and all variables are written as log-deviation of their steady-state. To solve for the *zero-order* portfolio, we need a first-order solution for such variables. The apparent problem is that these first-order solutions will depend on such *zero-order* portfolio. To see how, define the domestic country's net wealth as $W_t \equiv B_{H,t} + B_{F,t}$. Write the domestic budget constraint as:

$$W_t = R_t W_{t-1} + R^X_t B_{F,t-1} + \gamma \left[Q_t g^*(S_t)^{\theta-1} C^*_t - h(S_t)^{\theta-1} C_t\right] \tag{25}$$

Where $R^X_{t+1} \equiv e^{\psi_t} \frac{Q_{t+1}}{Q_t} R^*_t - R_t$ is the excess return on foreign bonds. Perform a first-order linear approximation around the steady state, using the fact that, due to symmetry $W_{ss} = 0$, and the fact that, at first-order, $q_t = (1 - 2\gamma)s_t$. Also recall that the steady-state for foreign bonds $B_{F,t-1}$ is the *zero-order* portfolio that we wish to endogenously determine, say B_F. Divide the result both sides by Y_{ss}:

$$w_t = \frac{1}{\beta} w_{t-1} + r^X_t b_F + \gamma \left(\frac{(2\theta(1-\gamma) - 1)}{1 - 2\gamma} q_t - c^R_t \right) \tag{26}$$

Where $w_t \equiv (W_t - W_{ss})/Y_{ss}$, $b_F = B_F/Y_{ss}$, $r^x_t = q_{t+1} - q_t + r^*_t - r_t + \psi_t$, $y^R_t = \log(Y_t) - \log(Y^*_t)$. Observe that now the relevant state variable is only the total wealth position[10],

[9]Under CRRA preferences. See **DEVEREUX and SUTHERLAND (2014)**

[10]This is not a property of the first-order approximation of equilibrium. In the non-linear version, we can define total wealth as the unique endogenous state variable, but policy functions must be consistent with such wealth for each state, increasing the number of equations in the system by the number of states.

w_{t-1}. A positive position on foreign bonds ($b_F > 0$) implies that the country will be wealthier when the excess returns on foreign bonds is positive, increasing its consumption in the next periods.

2.2 Solution

Observe that the time variation of the portfolio does not matter for an approximation of first-order solution. The only relevant term is the *zero-order* term in a Taylor series approximation of the true equilibrium portfolio function. Therefore, this method delivers a solution that exhausts all the macroeconomic implications of portfolio choice at this level of approximation.

The *zero-order* solution serves as an approximation of the mean of the ergotic distribution of portfolios when the volatilities of the shocks are small and the model is stationary. As mentioned, although the model will not be stationary, it preserves very similar properties to stationary versions of the model. Thus, we can look at the *zero-order* portfolio as an approximation of the *long-run* portfolio position, and extract some intuition. We begin reducing the linearized model as most as possible:

Lemma 1 *Let $\xi_t \equiv b_F r_t^X$ be a zero-mean shock, and assume that both countries' endowment processes are equal, but with different innovations. We can reduce the linearized model into a system of two equations and two variables:*

$$E_t q_{t+1} = q_t - \omega_1 y_t^R - \omega_2 \psi_t \tag{27}$$

$$w_t = \frac{1}{\beta} w_{t-1} + \mu q_t - \frac{\gamma}{1-2\gamma} y_t^R + \xi_t, \quad \textit{where} \tag{28}$$

$$\omega_1 \equiv \frac{(1-2\gamma)(1-\rho^y)\sigma}{4\sigma\theta\gamma(1-\gamma)+(1-2\gamma)^2} > 0, \quad \omega_2 \equiv \frac{(1-2\gamma)^2}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2} > 0, \quad \textit{and}$$

$$\mu \equiv \frac{2\gamma\theta(1-\gamma)-\gamma(1-2\gamma)}{(1-2\gamma)^2} > 0$$

Proof. See Appendix B.1

The spirit of this *lemma* is to arrange the model such that we would if $b_F r_t^X$ were an *i.i.d zero mean* shock. Actually, $E_t r_{t+1}^X = 0$ is an equilibrium condition, but the covariance structure of $b_F r_t^X$ is endogenous. However, we do not need covariance information for a first-order solution. Therefore, this *lemma* is an intermediate step to solve for first-order dynamics of consumption and exchange rate given a *zero-mean* surprise every period.

Proposition 1 *Assume that endowment shocks follow an AR(1) process with the same lag coefficient. The solution of real exchange rate consistent with the appropriated transversality condition, given the exogenous zero-mean shock ξ_t, is:*

$$q_t = \lambda_y y_t^R + \lambda_\psi \psi_t - \frac{1-\beta}{\mu}\left(\frac{1}{\beta} w_{t-1} + \xi_t\right), \quad \textit{where} \tag{29}$$

$$\lambda_y \equiv \left(\frac{\beta\sigma(1-2\gamma)(1-\rho^y)}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2} + \frac{(1-2\gamma)(1-\beta)}{2\theta(1-\gamma)-(1-2\gamma)}\right)\frac{1}{1-\beta\rho^y} > 0$$

$$\lambda_\psi \equiv \frac{(1-2\gamma)^2}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2}\frac{\beta}{1-\beta\rho^\psi} > 0$$

Proof. See Appendix B.2

The coefficient $\lambda_y > 0$ implies that an increase in relative output depreciates the real exchange rate. This occurs because higher relative output means a higher quantity of goods H relative to good F available in the economy. Therefore, the relative price of good H falls. Due to home bias ($\gamma < 1/2$), domestic consumption basket relative price also falls, or, in other words, the real exchange rate depreciates.

The coefficient λ_ψ is also a positive number. As shown in **ITSKHOKI and DMITRI, JPE 2021**, the calibration that corrects most of the exchange rate puzzle in the model includes $\beta\rho^\psi \to 1$, which implies $\lambda_\psi \to \infty$. This implies that even low fluctuations of ψ_t will cause high real exchange rate fluctuations, but also with high fluctuations of consumption.

The last can be avoided with a calibration $\gamma \to 0$, solving the exchange rate disconnect puzzle in the model, as shown by **ITSKHOKI and DMITRI, JPE 2021**. Both calibrations account for a solution for some puzzles such as high exchange rate volatility, exchange rate disconnect, low predictability, and UIP break.

Corollary 1 *Relative consumption $c_t^R \equiv c_t - c_t^*$ solution that is consistent with the appropriate transversality condition, given the exogenous zero-mean shock ξ_t, is given by:*

$$c_t^R = \Theta_1 y_t^R + \Theta_2 \left(\frac{1}{\beta} w_{t-1} + \xi_t \right) - \Theta_3 \psi_t \quad \textit{where} \tag{30}$$

$$\Theta_1 \equiv \frac{1 - 2\gamma - 4\theta\gamma(1-\gamma)\lambda_y}{(1-2\gamma)^2} > 0, \quad \textit{and}$$

$$\Theta_2 = \frac{4\theta(1-\gamma)(1-\beta)}{2\theta(1-\gamma) - (1-2\gamma)} > 0, \quad \textit{and}$$

$$\Theta_3 = \frac{4\theta\gamma\beta(1-\gamma)}{(1-\beta\rho^{\psi})(4\theta\sigma\gamma(1-\gamma) + (1-2\gamma)^2)} > 0$$

Proof. See Appendix B.3

From the consumption policy function, we see that a positive endowment shock is associated with higher consumption and a higher exchange rate. But a risk-premium shock is associated with lower consumption and higher exchange rate[11]. Shocks provide different incentives for the role of the foreign asset in the portfolio. Under the endowment shock, the foreign asset provides good remuneration when consumption increases. Under the risk-premium shock, the foreign asset provides excess returns when consumption decreases.

The reason that a risk-premium shock depreciates the exchange rate and drops consumption is through the UIP parity. Higher risk-premium shocks increase the return on foreign

[11]Again, if $\beta\rho^{\psi} \to 1$, Θ_3 may be a higher coefficient. But as shown by **ITSKHOKI and DMITRI, JPE 2021**, a necessary calibration that solves the exchange rate disconnect puzzle is $\gamma \to 0$. But in that case $\Theta_3 \to 0$. Therefore, a lower γ will compensate for a high $\beta\rho^{\psi}$.

bonds, which endogenously causes a decrease in both $E_t \Delta q_{t+1}$ and an increase in r_t to maintain such a condition. The decrease in $E_t \Delta q_{t+1}$ is achieved through a high increase in q_t, which is expected to decrease in the future. The increase in r_t is responsible to draw consumption down, through the elasticity of substitution in time. The increase in q_t also crowds out domestic consumption, because foreign households will switch expenditure toward the H good, which is cheaper, reinforcing the drop in domestic consumption.

Due to risk-aversion that arises from the second-order approximation of Euler equations, agents want assets that provide excess returns when consumption falls. In the frictionless model ($\psi_t = 0$), the foreign asset provides a hedge to consumption if the domestic country holds a short position or a (non-defaultable) debt. Intuitively, the country prefers to issue debt denominated in foreign currency rather than buy the asset[12], because when consumption falls, the cost of such debt also falls.

When only the financial friction is included ($\psi_t \neq 0$), even a low volatility will affect consumption if $\beta \rho^{\psi} \to 1$ if $\gamma > 0$. When this shock happens, a risk-averse agent will want to have a long position on the foreign asset, because exchange rate increases will provide excess returns when consumption drops. The exposition in such an asset will decrease as $\gamma \to 0$ because excess returns become much more noiser than consumption fluctuations. Clearly, when both shocks are present we will have some combination of effects, and volatilities will be relevant information for portfolio composition due market incompleteness.

We already characterize the law of motion for real exchange rates. We can proceed to characterize excesses returns as a function of the foreign position b_F:

Lemma 2 *Excesses returns on the foreign asset, given the (endogenous) position long-run*

[12] Recall that here we have perfect enforcement in financial markets. Therefore, a one-period short or long position are perfect substitute asset.

position b_F, is given by:

$$r^X_{t+1} = \frac{1}{1 + \frac{(1-\beta)}{\mu} b_F} \left[\lambda_y \xi^{y^R}_{t+1} + \lambda_\psi \xi^{\psi}_{t+1} \right]. \tag{31}$$

Proof. See Appendix B.4

If $|b_F|$ is not too large, we can use the heuristic $1 - \beta \approx 0$ to interpret the excess return equation. In general equilibrium, both shocks cause an excessive return on the foreign asset, but the endowment shock increases consumption and the risk-premium shock decreases consumption. In the next result, we formalize the intuition obtained through policy functions.

Proposition 2 *In the complete markets frictionless version of the model with $\psi_t = 0$, optimal (long-run) portfolio on foreign currency is:*

$$b^0_F = \frac{1 - 2\gamma + \sigma(1 - 2\theta(1-\gamma))}{\sigma(1-2\gamma)} \frac{\gamma}{1 - \beta\rho^y} \tag{32}$$

If there is home-bias ($\gamma < 1/2$), the domestic country takes a short position on the foreign asset, $b^0_F < 0$.

Proof. See Appendix B.5

The intuition for the result arises from the correlation between the endowment shock and consumption. When the (relative) endowment shock is positive, (relative) consumption increases and the real exchange rate depreciates. The real exchange rate depreciates due to the fact that there are more resources to build additional domestic bundles C_t than foreign bundles C^*_t, therefore the price of the latter will be higher. This effect occurs due to the presence of home-bias in consumption. With real exchange rates rising, excess returns on foreign assets are positive. Also, observe that endowment volatilities do not appear in the solution. This occurs due to the complete market structure without the risk-premium shock. Since households can allocate consumption in each linearly independent state of nature, volatility does not matter for portfolio position. Now we formalize portfolio allocations for the model with both shocks.

Proposition 3 *In the incomplete markets version of the stylized model with $\psi_t \neq 0$, the optimal (long-run) portfolio on foreign currency is:*

$$b_F = b_F^0 + \Omega \left(\frac{\beta}{1 - \beta\rho^{\psi}} \right)^2 \frac{\sigma_\psi^2}{\sigma_y^2}, \quad \textit{where} \tag{33}$$

$$\Omega = \frac{(2\theta\gamma(1-\gamma) - \gamma(1-2\gamma))(1-2\gamma)}{\sigma\lambda_y(1-\beta)(4\sigma\theta\gamma(1-\gamma) + (1-2\gamma)^2)} > 0$$

Under the calibration $\beta\rho^{\psi} \to 1$, the domestic country takes a long position on the foreign asset, $b_F > 0$.

Proof. See Appendix B.5.

When $\psi_t \neq 0$, foreign bonds pay excess returns that correlate negatively with the risk-premium shock, raising demand for the asset from domestic agents that wish to hedge against negative risk-premium shock. The optimal long-run portfolio for the domestic agent now can be decomposed in two terms: the (negative) zero-volatility portfolio b_F^0 that is optimal when the agent only faces endowment risk, and a positive term that is higher whenever the risk-premium shock is more persistent ($\beta\rho^{\psi} \to 1$) or more volatile (higher σ_ψ). If the risk-premium shock is persistent enough, then even low risk-premium shocks will induce large, and almost permanent, changes in the real exchange rate. This induces a persistent decrease in consumption and a large increase in the excess return of the foreign asset. Since the domestic country is risk-averse, a positive position on such a bond will guarantee a large one-time increase in wealth, which will soften the impact of the shock on consumption across time.

As mentioned earlier and now formalizing, as $\gamma \to 0$, we have that $b_F \to b_F^0$ because exchange rate high fluctuations are not reflected anymore in consumption fluctuations. Therefore, a risk-averse agent will not want high expositions in such an asset due to the high volatility that it will be associated with, even though that, on average, may provide a good hedge. A calibration $\gamma = 0$ is also not good because it will still cause a version of the Backus-Smith puzzle where the exchange rate is not correlated at all with macro fundamentals. Because we want this correlation to be low and negative in the model, a positive γ but near zero is ideal, as explained in detail by **ITSKHOKI and DMITRI, JPE 2021**. We show here that this same calibration that corrects exchange rate puzzles also reproduces positive positions on the foreign asset.

2.3 Calibration and Numerical Results

Intertemporal discount $\beta = 0.99$ is set to a quarterly frequency. The risk-aversion $\sigma = 2$ is standard. The degree of substitution between home and foreign goods is set to $\theta = 1.5$. This is the most contested calibration, it follows the estimates of **Feenstra, Luck, Obstfeld, and Russ (2014**. The persistence and volatility of the endowment fluctuations are set accordingly in Brazil's real GDP growth[13]: $\rho^Y = 0.87$, $\sigma_Y = 0.063$. Although calibrated using a specific country, these values are consistent with most quarterly endowment or productivity shock calibrations. Concerning home-bias, we set $\gamma = 0.05$. This parameter is typically calibrated using the average imports over GDP. Under this argument, this may be an unusual calibration, which was expected to be around 15% or 20% however, as shown by **Oleg and Dmitri (2021)**, when $\gamma \to 0$ and $\rho^\psi \to 1$ we observe the solution of some common exchange rate puzzles. One very popular in the literature is the exchange rate disconnect puzzle, which is referred to as the similar level of fluctuations between exchange rate and other macro fundamentals that occurs in traditional models, but in the data, exchange rates appear to be much more volatile and accompanied by much smaller movements in consumption and income. Therefore, the calibration is set to reproduce the most possible realistic exchange

[13]Here we take monthly Brazil's GDP, use the inflation index IPCA to deflate the series, and transform it to quarters, summing up every three months. After, we remove the trend using a quadratic regression in time. Finally, we estimate an AR(1) without constant and obtain the calibrated values.

rate in the model. We still can calibrate $\gamma = 0.05$ and have higher imports over GDP if we consider an endogenous production model, where firms use imported units as input, as it is considered in the extensions section. We follow **Oleg and Dmitri (2021)** and set $\rho^{\psi} = 0.97$. Table 1 resumes the calibration, and figure 1 illustrates the results for different calibrations of the financial shock volatility, σ_{ψ}.

Description	Parameter	Calibration
Intertemporal Discount Factor	β	0.99
Risk-aversion	σ	2
Substitution Degree between H and F goods	θ	1.5
Endowment Persistence	ρ^{Y}	0.87
Endowment Volatility	σ_{Y}	0.063
Home-Bias	γ	0.05
Financial Shock Persistence	ρ^{ψ}	0.97

Table 1: Calibration Baseline Model

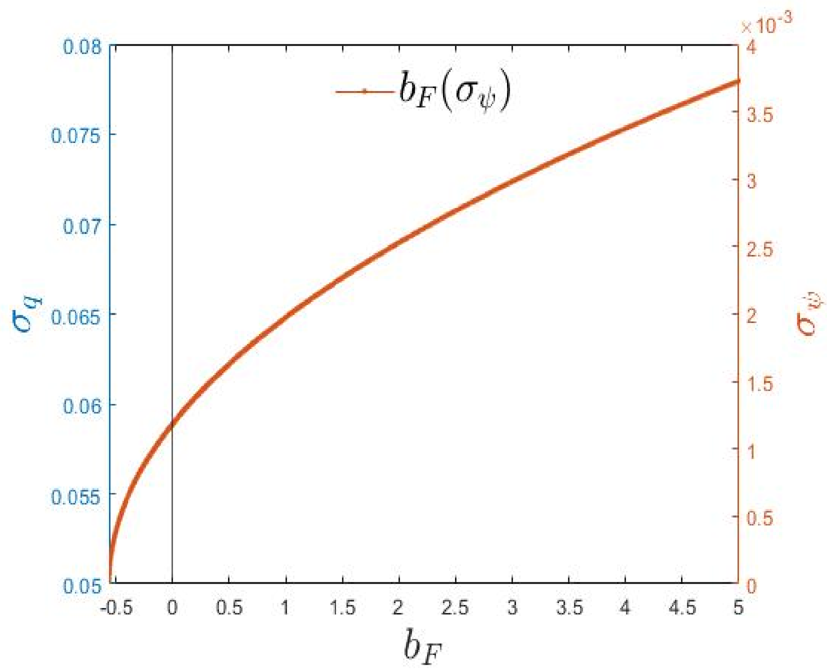


Figure 1: Optimal Portfolios and Exchange Rate Volatility

Without the risk-premium shock ($\sigma_{\psi} = 0$), the economy holds on average 25% of annual GDP of debt denominated in foreign currency. The country chooses to hold debt instead of assets because in bad times (low y_t^R) real exchange rates appreciate, and the service of debt decreases because foreign currency becomes cheaper. This decrease in the bad state is much appreciated by the home economy because consumption falls in this bad state. So a risk-averse country wishes to hold debt in foreign currency when exchange rates are mainly driven by endowment or productivity shocks.

Now, as the volatility of the risk-premium shock increases, the risk of a bad risk-premium state increases, and such a bad state is expected to last long a time due to the high calibration of $\rho^{\psi} = 0.97$. In such a state, consumption falls and the exchange rate increases, therefore a risk-averse country wishes to hold a positive position of such assets as a hedge for such a state. The previous intuition for endowment shocks is still there, so the volatility of the risk-premium shock must be high enough to overcompensate for the correlation caused by such shock.

Positive levels of reserves already appear with significantly low values of risk-premium volatility. With $\sigma_{\psi} = 0.3\%$, the country wishes to hold 75% of annual GDP on reserves. One can see that the model can reproduce the reserves and exchange rate volatility consistent with the empirical evidence. For example, if we set $\sigma_{\psi} = 0.2\%$, the resulting equilibrium will be reserves at 24% of annual GDP and exchange rate volatility around 6.5%, which is similar to Brazil's values[14]. The reason why such small volatilities already induce these high values is that the financial risk-premium shocks matter a lot for the linearized system. The microfoundation of the risk-premium shock will induce a much smaller coefficient on the linearized system, allowing more considerable volatilities.

[14] Around 25% of reserves over GDP and 8% of exchange rate volatility.

3 Microfoundation of the Financial Friction

We have shown that the frictionless with only endowment shocks generates *long-run* negative position on foreign currency. Countries wish to hold *non-defaultable* debt denominated in foreign currency, instead of reserves. The reason is that higher consumption is associated with more depreciated real exchange rates[15]. This relation comes from the general equilibrium pricing of goods H and F. Due to home bias, higher domestic endowment means that there are more resources to build C_t baskets than C_t^*, decreasing the price of the first, *e.g.* a real exchange rate depreciation, although consumption also rises. Therefore, when consumption is low, real exchange rates are lower and the service of debt is also lower. The negative position on foreign bonds softens consumption drops in bad states.

We extend the model with a reduced form of financial friction and showed that, under strategic calibration, this financial friction can account for a positive *long-run* position on foreign currency. Strategic calibration refers to a calibration that corrects most of the common exchange rate puzzles that appears in open macro models, as shown by **Oleg and Dmitri (2021)**. The reason is that such financial friction is a source of a negative correlation between consumption and real exchange rates, a correlation much more consistent with data, especially for emerging markets. The *risk-premium* shock induces higher returns on the foreign bond, demanding higher levels of expected appreciation on foreign currency or higher domestic interest rates. This is achieved with a contemporaneous one-time increase of real exchange and domestic interest rates, which tends to decrease in expectation in the future. The increase in domestic interest rates is consistent with a decrease in domestic consumption. Households spot higher domestic rates, meaning that consuming today is expensive. This shock causes, through general equilibrium, contemporaneous exchange rate

[15] This is actually a general feature of complete market models. As it is known that, in the model with full *Arrow-Debreu* securities, the linearized condition $\sigma c_t^R = q_t$ emerges. Under such an equation, consumption will always be positively correlated with the real exchange rate, no matter the shock. Therefore, an incomplete market is a necessary condition to reproduce more realistic real exchange rates and, consequently, reserves.

depreciation, higher interest rates, and lower consumption, a convenient interpretation for a *risk-premium* shock.

In this section we provide a microfoundation of such shock. The microfoundation is a version of **ITSKHOKI and DMITRI, (2021)**, which is based on the noise trader and limits-to-arbitrage model of **De Long, Shleifer, Summers, and Waldmann (1990)** and its adaptation to the exchange rate market by **Jeanne and Rose (2002)**. This is a version here the domestic country having access to foreign bonds, but only accommodated by households and chosen by the central bank through a rule. In this type of financial sector model, it is necessary to restrict the access of domestic households to foreign bonds, if one wishes to approximate the equilibrium at first-order. The reason is that financial intermediaries demand an excess return on foreign bonds, but if households are free to arbitrate between bonds they would do so using such increased return fully allocating in foreign bonds, since at first order they only care for expected return. In practical terms, two linear equations appear in the system that is impossible to simultaneously satisfied.

This microfoundation is still subject to such limitations, but since we are interested in observing the country portfolios, we need some flexibility to allow country allocations. We proceed making two further simplifications. The first is to indeed restrict households' access to foreign bonds, but make them internalize some bond allocation in their budget constraint[16]. This bond position is taken as given by households and is chosen by a "rule" imposed by the monetary authority, which is a second-order approximation of euler equations. The second is for the tractability of closed-form solutions, which is restricting households to form expectations of excess returns that would arise in the face of financial intermediaries' demand.

[16]Although this is not necessary for the nonlinear models, because household euler equations and financial intermediaries' bond demand could both be satisfied, it is necessary for the first-order approximated model, as will it be soon clearer

3.1 Financial Sector

The "rule" responsible to form portfolios is the same second-order approximation of the Euler equation 24, so it is an optimal rule in the light of Euler equations, respecting household preferences and risk-aversion. In this microfoundation, there are *noise-traders* that exogenously demand foreign bonds. The word *noise* comes from the fact that such demand does not depend directly on the country fundamentals, captured by c_t^R, w_t, or q_t. Therefore, their demands (short or long) are purely viewed as shocks in the light of the model[17]. Domestic country euler equation now is:

$$E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}R^*_{t+1}\frac{Q_{t+1}}{Q_t}\right]=1 \tag{34}$$

There are four types of agents operating in financial markets for each bond. Two of them were already included in the baseline model, which represents each country's demand for each bond. Those are captured by the bond market clear equations 18- 19. Now we include a *zero*-capital noise trade of mass $0<n<1$, demanding N_t of the domestic bond and N_t^* of the foreign bond. The *zero*-capital means that any short position on one bond must be financed by a long position on the other bond:

$$N_t=-N_t^* \tag{35}$$

Where, as usual, both quantities are denominated in terms of domestic consumption bundle, C_t. As mentioned, we assume that noise traders demand for foreign bonds is exogenous and given by a shock:

$$N_t^*=n\left(e^{\psi_t}-1\right) \tag{36}$$

[17] An alternative intuition is of a *noisy country.* Imagine that policymakers keep making public statements that are not consistent with welfare-improving behavior. Even though these statements are not made in practice, through breaks of euler equations or imposing additional frictions, traders become early anxious and demand foreign bonds to zero the exposition on the domestic country.

The last type of agents are financial intermediaries of mass $0 < m < 1$. They are responsible to intermediate both countries and noise trader demand for bonds. Let D_t and D_t^* be the amount of domestic bond and foreign bond held by the intermediary, respectively. Both quantities are denominated in terms of domestic consumption bundle C_t. Financial intermediaries are also *zero*-capital based:

$$D_t = -D_t^* \tag{37}$$

Since they intermediate the demand for bonds, the total demand for each bond must be equal to the total bond supplied by the financial intermediary. Already imposing the zero capital position assumption, bond market clear now becomes:

$$B_{H,t} + B_{H.t}^* + N_t = -D_t \tag{38}$$

$$B_{F,t} + B_{F,t}^* - N_t = D_t \tag{39}$$

Each financial intermediary in the mass $[0, m]$ chooses the amount of foreign bonds to intermediate, d_t^*. Because of the *zero*-capital position, each foreign bond intermediated yields a return of $R_{t+1}^X = \frac{Q_{t+1}}{Q_t} R_t^* - R_t$. The position d_t^* is chosen in order to maximize the following mean-variance utility function:

$$\max_{d_t^*} E_t R_{t+1}^X d_t^* - \frac{\omega}{2} \text{var}_t \left(R_{t+1}^X \right) d_t^{*2}$$

,

Where ω is a risk-aversion parameter of the mean-variance agent. Since the financial intermediary is infinitely small in the continuum, it does not internalize her position impact on R_{t+1}^X, and it takes as a given process. Aggregating the individual solution we have:

$$D_t^* = m\frac{E_t R_{t+1}^X}{\omega \text{var}_t \left(R_{t+1}^X\right)} \tag{40}$$

3.2 Equilibrium

Definition 2 *An equilibrium are functions* $\{C_t, C_t^*, Q_t, S_t, B_{H,t}, B_{H,t}^*, B_{F,t}, B_{F,t}^*, R_t, R_t^*, D_t, N_t\}$ *defined over the states* $(Y_t, Y_t^*, \psi_t, B_{H,t-1}, B_{F,t-1})$ *such that satifies equations 2, 14, 15, 17, 38, 39, 20, 34, 22, 23 and now, additionally, 36 and 40. Those equations correspond to 1 terms of trade equation, 2 goods market-clear, 2 bonds market-clear, 4 Euler equations, 1 budget constraint, 1 noise trader demand, and 1 intermediary demand.*

Equilibrium definition now consists of two additional variables, N_t, D_t and two additional equations. The noise trader variable and equation are actually trivial and can be omitted from the system. The intermediation quantity D_t is important since it will affect household budget constraints through bond demands.

A positive noise trader shock will increase the amount of intermediation required from the financial intermediaries. This can be achieved through two channels. The first is the simple increase of the bond supply from the intermediary. It must be accompanied by high expected excess returns to compensate for the larger position. The second is a *crowd out* of private demand for bonds. The total demand for the bond after the noise trader shock may be too high which would induce large drops in consumption, through the necessary exchange rates to accommodate the necessary excess return asked by financial intermediaries. Households may wish to reduce the demand for such bonds to accommodate some noise trader demand and avoid a larger impact on exchange rates and consumption.

3.3 Solution

Solving this model implies finding policy functions defined over the state variables that are consistent with the model equations, for any point in the state space. This is not a trivial task due to the high nonlinearity of the system. Therefore, to acquire a microfounded version of our baselin linearized system, and obtain a *pen and paper* solution, we make some simplifications of the model equations.

We suppose that households do not choose the foreign bonds portfolio, but just accommodate a decision made by the central bank. The central bank chooses a portfolio that is consistent with a second order approximation of euler equation, given by 24:

$$E_t \left(\sigma c^R_{t+1} - q_{t+1}\right) r^X_{t+1} = 0$$

Household takes the position as given, and the decision impacts their consumption and wealth but is taken as a zero-mean exogenous shock received by households. This is consistent with the first-step solution portfolios as in 1. An additional simplification is that all reserves must be financed by domestic debt, and not by financial intermediaries. This is consistent with the idea that a central bank is choosing the reserves and households are just internalizing them because reserves may be a large pool of resources, and, at least in the *long-run*, it must be financed by other agents that are not mere financial intermediaries.

The necessity of this simplification is technical, due to the non-existence of first-order approximated equilibria. To see this, note that the financial shock impacts the system through the bonds market clearing, and not through a UIP condition.

$$B_{H,t} + B^*_{H,t} + n(e^{\psi_t} - 1) + m\frac{E_t(R^X_{t+1})}{\omega \text{var}_t(R^X_{t+1})} = 0. \tag{41}$$

The modified UIP appears after linearization of such equation, using the intermediation position and noise trader process:

$$r_{t+1} - r^*_{t+1} - \mathbb{E}_t(\Delta q_{t+1}) = \chi_1 \psi_t - \chi_2 b_t \tag{42}$$

Where $\chi_1 = \frac{n\omega\sigma_q^2}{m}$, and $\chi_2 = \frac{\omega\sigma_q^2}{m}$, and b_t is the total demand for domestic assets $B_t \equiv B_{H,t} + B^*_{H,t}$, over steady-state GDP. The parameter $\sigma_q \equiv \text{var}_t \Delta q_{t+1}$ is endogenous but taken as given by the financial intermediary, because the financial intermediary is a point in a continuum of mass $[0, m]$, therefore it does not internalize their impact on the exchange rate process. The term $\text{var}_t \Delta q_{t+1}$ is constant in time, due to a general property of linear processes.

A positive noise trader demand shock ψ_t must be accommodated by increasing expected returns from the intermediary position, or by crowding out domestic-denominated debt through a reduction in b_t. Now, combine the Euler equations from the domestic and foreign household utility maximization problem to obtain:

$$E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} R^X_{t+1} \right] = 0 \tag{43}$$

$$E_t \left[\beta \left(\frac{C^*_{t+1}}{C^*_t} \right)^{-\sigma} \frac{Q_{t+1}}{Q_t} R^X_{t+1} \right] = 0 \tag{44}$$

These are pricing equations for the excess return on bonds and are a necessary condition for an existence of an interior solution for debt quantities $B_{H,t}$, $B_{F,t}$, $B^*_{H,t}$ and $B^*_{F,t}$. A violation of any of these conditions will generate an infinite demand for one bond, which will be financed by a corresponding infinite short position on the other, and no equilibrium will be possible. Note that equations (41), (43) and (44) are consistent: the former constrains the path for the expected excess returns $E_t[R^X_{t+1}]$ while the latter prices the excess returns path according to the agents pricing kernel.

However, linearizing equations (43) and (44) and combining results in a risk-neutral version of the pricing equation, which corresponds to the conventional uncovered interest parity (UIP) condition:

$$r_t - r_t^* - \mathbb{E}_t(\Delta q_{t+1}) = 0. \tag{45}$$

While the non-linear versions (41), (43) and (44) are consistent, its linear counterparts (42) and (45) are no longer consistent, since one equation implies an UIP deviation while the other does not. In the non-linear analog of equation (45), we have $E_t R_{t+1}^X = 0$, which is inconsistent with an equilibrium since both intermediaries and households would not demand a non-zero leveraged position on bonds that would be necessary to finance noise traders, due to risk-aversion. Linearization removes the risk-aversion component of the portfolio selection, effectively muting a crucial dynamic for determining the position of agents on assets, which is out goal.

To solve this issue, we remove the linearized versions of the Euler equations that emerge from the maximization problem over foreign bonds for the domestic and foreign households, so that equation (45) is no longer part of the model. Their remotion can be interpreted as households not choosing portfolios on foreign bonds, but just internalizing some given amount in their budget constraint. The quantity chosen will be set by the Central Bank that follows a "rule", which is a non-linear version of Euler equations, given by equation 24. Since we have two fewer equations on the model, we need to impose additional constraints on the linearized system so that it can be solved. This is done by restricting $b_t = b_t^* = 0$, which states that one country should finance the position of the other. It has a straightforward interpretation: a country only holds bonds that were written by the other country, and not by financial intermediaries. This assumption is natural, since in the long run, as noise trader shocks dissipate, equation (41) (or its linear analog (42)) imply that $B_t \to 0$, and if our goal is to determine the steady-state position on foreign bonds by the domestic household, this steady-state position must be consistent with a zero total domestic debt $B = 0$.

We need an additional simplification. Although there is an expected excess return on foreign bonds in the model, this is not internalized by households in their budget constraint. This is necessary to make a closed-form solution tractable and to correctly apply the method of **Devereux and Sutherland (2014)**. An interpretation is that since exchange rates will have a nearly indistinguishable behavior from a random walk[18], at least in finite samples, households don't expect excess returns over it.

The remaining equations of the system are unchanged, it is straightforward to establish the following Lemma, analogous to Lemma 1:

Lemma 3 *Let $\xi_t \equiv b_F r_t^X$ be a zero-mean shock, and assume that both countries' endowment processes are equal, but with different innovations. We can reduce the linearized model, with the additional assumptions of $b_t = b_t^* = 0$ and ξ_t as a zero-mean shock, into a system of two equations and two variables:*

$$E_t q_{t+1} = q_t - \omega_1 y_t^R - \hat{\omega}_2 \psi_t \tag{46}$$

$$w_t = \frac{1}{\beta} w_{t-1} + \mu q_t - \frac{\gamma}{1-2\gamma} y_t^R + \xi_t, \quad \textit{where} \tag{47}$$

$$\omega_1 \equiv \frac{(1-2\gamma)(1-\rho^y)\sigma}{4\sigma\theta\gamma(1-\gamma) + (1-2\gamma)^2} > 0, \quad \hat{\omega}_2 \equiv \frac{n\omega\sigma_q^2}{m}\omega_2, \quad \textit{and}$$

$$\mu \equiv \frac{2\gamma\theta(1-\gamma) - \gamma(1-2\gamma)}{(1-2\gamma)^2} > 0$$

Proof. Shown in the appendix

The linear system is nearly identical to the previous model, except that the parameter $\hat{\omega}_2$ that multiplies the noise trader shock is now dependent on the ratio of the measure of noise traders and of financial intermediaries, n/m, the intermediaries risk aversion level ω and the (endogenous) volatility of the exchange rate, σ_q^2. As a consequence, we can establish the main result:

[18] See **Oleg and ItshKhoki (2021)** to see how the calibration generates this behavior.

Proposition 4 *In the incomplete markets version of the model with microfoundation for the financial shock, the optimal (long-run) portfolio on foreign currency is:*

$$\hat{b}_F = b_F^0 + \Omega \left(\frac{\beta}{1 - \beta\rho^{\psi}} \right)^2 \left(\frac{n\omega}{m} \right)^2 \frac{\sigma_{\psi}^2}{\sigma_y^2} \sigma_q^4, \quad \textit{where} \tag{48}$$

$$\Omega = \frac{(2\theta\gamma(1-\gamma) - \gamma(1-2\gamma))(1-2\gamma)}{\sigma\lambda_y(1-\beta)(4\sigma\theta\gamma(1-\gamma) + (1-2\gamma)^2)} > 0$$

Under the calibration $\beta\rho^{\psi} \to 1$, the domestic country takes a long position on the foreign asset, $\hat{b}_F > 0$.

Proof. Shown in the appendix.

Persistent risk-premium shocks that are associated with an almost permanent drop in consumption induce agents to hold reserves, which pay a greater real return in such events due to real exchange rate depreciation. The intuition now is analogous: noise trader persistent demand for a long position on foreign bonds causes a depreciation of domestic currency which is associated with a permanent drop in consumption. This movement induces agents to hold reserves, to hedge against the reduction on consumption. Demand for reserves will be higher the larger the ratio of noise traders to financial intermediaries n/m grows, and as the risk aversion coefficient of financial intermediaries ω becomes larger.

While the above intuition is valid, the model must be closed by determining the equilibrium conditional variance for the real exchange rate $\sigma_q^2 = Var_t(\Delta q_{t+1})$, which is constant at any point in time[19]. Since the equilibrium level for σ_q^2 will also depend on the level of reserves on the steady-state, we obtain a system of equations that determine an unique pair for $\hat{b}_F$ and σ_q^2 that is consistent with the equilibrium dynamical system.

[19]But is different from the unconditional variance of exchange rates depreciation. This is a property of linear stochastic processes. To see this, let $x_t \sim AR(1)$ stationary, then $var_t x_{t+1} = \sigma_{\epsilon}^2$, while $var x_{t+1} = \frac{1}{1-\rho}\sigma_{\epsilon}^2$, where σ_{ϵ}^2 is the variance of the $i.i.d$ innovation. Both values are constant in time, but different while conditioning or not.

Proposition 5 *The ex-post solution for the real exchange rate growth in the microfounded linearized system is given by:*

$$\Delta q_{t+1} = -\Sigma_1 y_t^R - \hat{\omega}_2 \psi_t + \lambda_y \xi_{t+1}^y + \lambda_\psi \xi_{t+1}^\psi - \frac{1-\beta}{\mu} \xi_{t+1}$$

$$\textit{where} \quad \Sigma_1 \equiv \frac{\sigma(1-2\gamma)(1-\rho^y)}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2} > 0.$$

This solution implies a constant conditional variance $Var_t(\Delta q_{t+1}) = \sigma_q^2$ with a unique stable solution given by:

$$\sigma_q^2 = \frac{1}{2\left(\frac{n\omega}{m}\right)^2 \lambda_\psi^2 \sigma_\psi^2} \left(1 + \frac{1-\beta}{\mu}\hat{b}_F\right)^2 \left[1 - \sqrt{1 - 4\lambda_y^2\sigma_y^2 \left(\frac{n\omega}{m}\right)^2 \lambda_\psi^2 \sigma_\psi^2 \left(1 + \frac{1-\beta}{\mu}\hat{b}_F\right)^{-4}}\right] \tag{49}$$

The system made of equations (48) and (49) for σ_q^2 and $\hat{b}_F$ contains an unique solution, such that $\sigma_q^2 > 0$.

Proof. Shown in the appendix.

Definition 3 *Let $b_F(\sigma_q^2)$ from equation 48 be the optimal demand of Central Bank for foreign assets given the volatility of exchange rate perceived by financial intermediaries, and $\sigma_q^2(b_F)$ from equation 5 the equilibrium exchange rate volatility given the Central Bank portfolio on the foreign asset. A general equilibrium consists of the pair (b_F^*, σ_q^{2*}) such that $b_F(\sigma_q^{2*}) = b_F^*$ and $\sigma_q^2(b_F^*) = \sigma_q^{2*}$.*

One can see from equation 5 that, depending on the calibration and on the value of foreign assets portfolio b_F, the square root of such number may yield a complex value. In such case, for such portfolio b_F, a general equilibrium does not exist for such approximation. This is especially the case when the parameters of the financial friction are high enough.

3.4 Calibration and Numerical Analysis

Here we provide a calibration for the microfounded model. This version of the model can allow higher risk-premium shocks without assigning high values to the portfolios. We can see why this happens when we look at the linearized system. Recall that ω_2 is the coefficient that multiplies the risk-premium shock in the baseline model:

$$\omega_2 = \frac{(1-2\gamma)^2}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2}$$

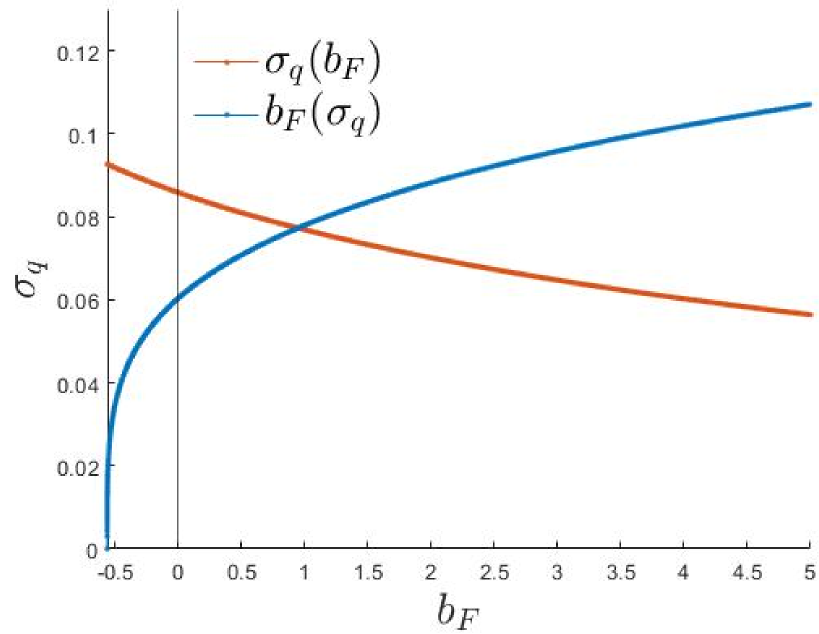
Now, when we look at the linearized system with the microfoundation of the risk-premium shock, we see that $\hat{\omega}_2 \equiv \frac{n\omega\sigma_q^2}{m}\omega_2$. Since, in equilibrium, real exchange rate volatility is a low value, then $\hat{\omega}_2 << \omega_2$, meaning that the risk-premium shock is quantitatively less relevant to the system. This allows us to tune much higher volatilities to the ψ_t shock.

The calibration of the previous parameters is the same. We set $\sigma_\psi = 0.05$. In this calibration, around 65% of noise trader's demands increase are around 5%. We still have three important parameters to be calibrated. The noise traders mass n, financial intermediaries mass m and their aversion ω. What actually matters is the value of their combination $n\omega/m$. We set $m = 1$ for normalization. We choose $\omega = 5$, a value that is considered high risk-aversion in typical efficient frontier problems. The intuition is that we're analyzing what was to be considered emerging markets, therefore financial intermediaries are still very risk-averse. Finally, we set $n = 1.3$ to reproduce an equilibrium level of foreign reserves of 24% of annual GDP, consistent with Brazil's current holdings. Table 2 summarizes the calibration and figure 2 shows the results. We can mixture more values between σ_ψ and $n\omega/m$ and still get similar results. The remaining parameters are the same.

The orange line illustrates the resulting equilibrium of exchange rate volatility, given a portfolio of foreign assets, given by equation 49. What this curve states is that, under the financial friction structure, higher foreign assets decrease exchange rate volatility. That is,

Description	Parameter	Calibration
Noise traders volatility	σ_ψ	0.05
Mass of Financial Intermediary	m	1
Risk-aversion of Financial Intermediary	ω	5
Mass of noise traders	n	1.3

Table 2: Calibration Microfounded Model



-

Figure 2: Equilibrium Foreign Assets and Exchange Rate Volatility

countries with higher reserves will present lower exchange rate volatility, a feature consistent with empirical evidence and endogenously generated in the model[20]. The blue line states the Central Bank optimal demand for foreign assets, given an exchange rate volatility perceived by financial intermediaries. Higher exchange rate volatilities perceived by financial intermediaries cause higher demand for reserves. This occurs because the financial shock becomes more relevant to the model, increasing the desire for hedge through foreign assets.

[20]Although this effect may be stronger in the empirical evidence due to the fact that Central Bank actively interferes in exchange rate markets for such purpose. A fact not captured in the model, but highlighted that such intervention is not necessary for long-run volatilities.

The resulting general equilibrium is the pair (σ_q^*, b_F^*) such that $b_F(\sigma_q^*) = b_F^*$ and $\sigma_q(b_F^*) = \sigma_q^*$, which is the point where these curves crosses. This can be viewed as a Nash equilibrium, where one player is a central bank choosing reserves, and the other player is the financial market delivering exchange rate volatilities, and the curves are each player's best response given the other player's strategy. In the calibration, the equilibrium is given by $(b_F^*, \sigma_q^*) = (0.95, 0.077)$, that is, an amount of 24% of annual GDP of reserves and 7.7% of exchange rate volatilities, both values consistent with Brazil's data.

It is also possible to see the stability of such equilibrium if we imagine a phase diagram in this figure. Say that we are currently out of equilibrium with the current exchange rate volatility actually higher, such as 10%. Then, the optimal policy for reserves is to increase, but as reserves increase, the resulting exchange rate volatility in equilibrium decreases. As the volatility decreases, reserves demand decreases, until the point where the resulting exchange rate volatility is consistent with the demand for reserves[21].

We also compare equilibrium portfolios and exchange rate volatility when changing some important parameters calibration. This is often called a sensitivity analysis, which consists of checking the results changes when we change the calibration. Figure 4 shows the results increasing both financial friction parameters and other more traditional parameters.

When we increase any of the parameters related to the size of the financial friction, we observe the same movement of both curves and an increase in foreign assets holdings with slightly the same exchange rate volatility. This occurs due to two movements. The first is that with higher relevance of the financial friction shock in the dynamics of the variables, agents find it optimal to hold more international reserves given the level of exchange rate

[21] We have to remember that this is an analysis of the stability of such steady-state values, and not a transition dynamics analysis. We can compare different steady-states, but in this approximation, we can't obtain transition dynamics.

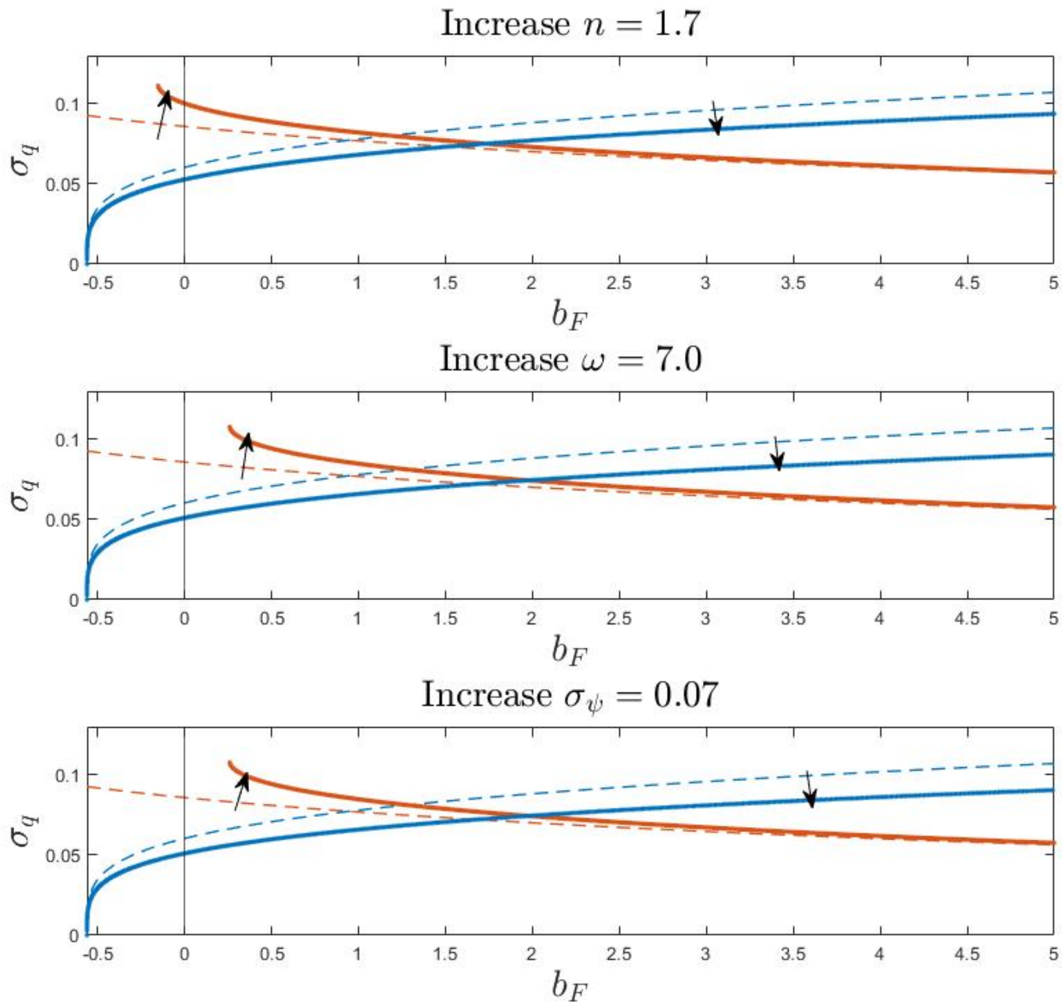


Figure 3: Sensitivity Analysis - Financial Friction

volatility perceived by the financial intermediary. This corresponds to a shift to the right of foreign asset demand. The other movement is that, given the same level of foreign assets, the equilibrium volatility of the exchange rate increases, due to the higher relevance of the financial shock. We can interpret both of these movements as the monetary authority buying more foreign assets to contain the increase in foreign exchange rate volatility.

Although the equilibrium results are not so different for these more intense calibrations of the financial shock, note that the orange line starts to appear only in some parts of the state space. This corresponds to the nonexistence of equilibrium for exchange rate volatility at negative (or positively small) levels for foreign assets. This can be interpreted as the Central Bank being unable to maintain exchange rate expectations anchored by the financial intermediaries. That is, the level of reserves is so small or the level of debt is so high that financial intermediaries will always increase their expected volatility for the real exchange rate, such that the equilibrium exchange rate volatility will always be higher than their expectation. But when foreign assets increase, Central Bank can achieve stability. If we set the financial friction to be strong enough, we may even not have the existence of an equilibrium[22]. The remaining parameters are more structural and deeper, hiding a lot of complexity behind them, therefore is natural to expect more relative importance for the general equilibrium.

When we increase σ, we increase both risk-aversion but also decrease the elasticity of substitution of consumption in time. The first effect corresponds to lower exposition to risk, and the last corresponds to lower incentive in saving consumption in favor of future consumption. The risk-aversion effect is consistent with the upward shift of the blue curve, which is less exposition to the risky asset and is also consistent with less consumption smoothing through savings. The shift upward of the orange curve corresponds to the lesser elasticity of substitution in time. That is, with the same amount of assets, agents are willing to consume more today, giving more volatility to exchange rates. The achieved equilibrium is a similar level of reserves but with more volatile exchange rates. The reason for increased equilibrium volatility is mainly due to the decrease in elasticity of consumption in time, but asset holdings remain the same due to higher risk aversion.

[22]This corresponds to the case where the downward slope line starts to appear only after the upward slope line.

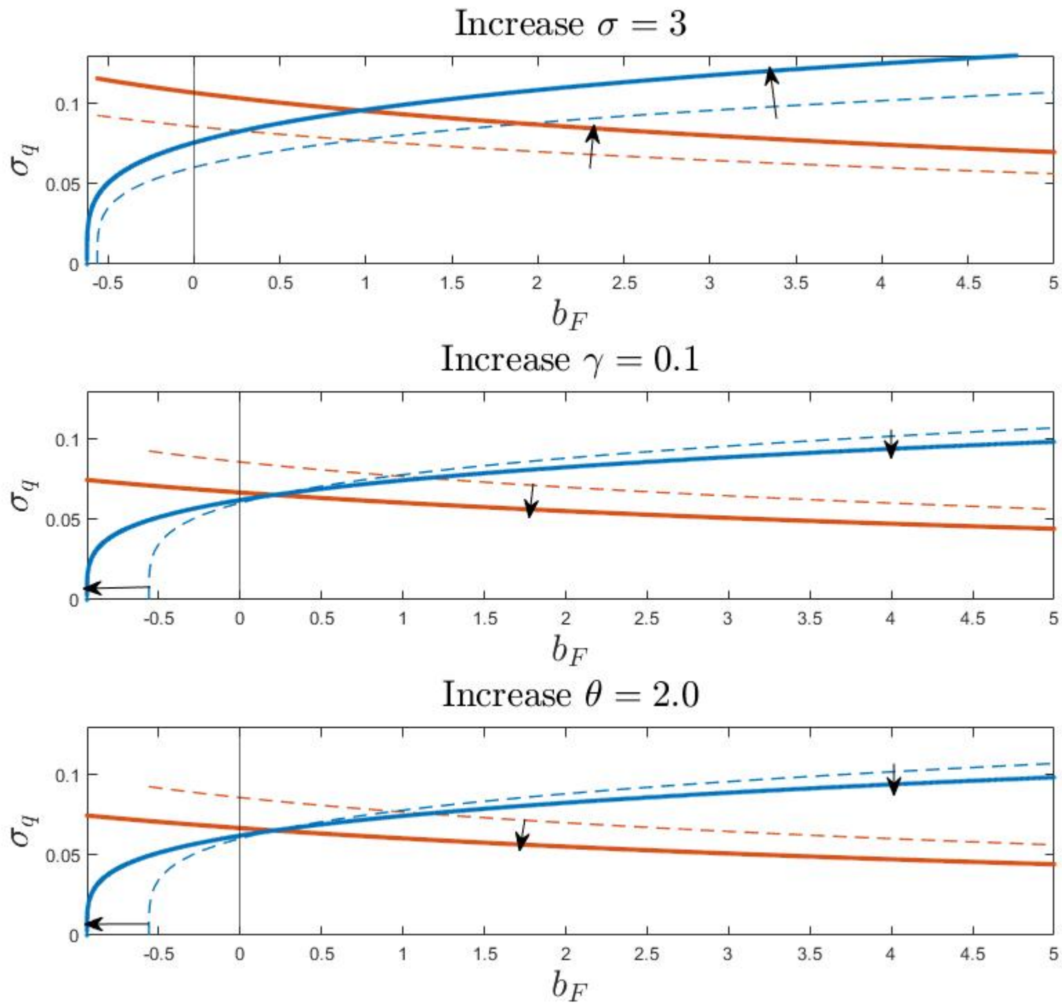


Figure 4: Sensitivity Analysis - Deep Parameters

When we increase γ, as shown by **Oleg and Dmitri (2021)**, the exchange rate disconnect puzzle starts to appear again in the model. Exchange rate fluctuations now are more associated with consumption fluctuations, meaning that when the exchange rate changes, consumption changes by more similar proportions. The optimal behavior for Central Bank is to increase the exposition to the asset because large movements in the asset price will be associated with larger movements in consumption. In the frictionless world, the demand for external debt is higher, and, when the friction is sufficiently relevant to hold reserves, the amount of reserves demanded by the country is higher for each unit of exchange rate volatility, explaining the movement of the blue curve. The orange also relates to such a puzzle. Because exchange rate volatility is more connected to macro fundamentals, the central bank loses the ability to control exchange rate volatility with foreign asset holdings, and the orange curve becomes flatter. The reason is that the driver for exchange rates is now more attributed to endowment shocks rather than financial shocks, weakening the hedging power of portfolios. Therefore, the orange curve is shifted down because endowment shock has a present value effect way lower than the financial shock, decreasing exchange rate volatility.

A very similar feature can be observed when θ increases. When θ increases, goods H and F are closer to substitutes, meaning that households can change one for another when the prices change. If the price of the good F increases, a household can rapidly substitute some amount of good F with good H, which is cheaper, making the cost of building the C_t basket only slightly increase. Therefore, higher θ is associated with less volatile exchange rates, shifting the orange curve down. Because this mechanism implies less volatile exchange rates[23], Central Bank wishes to expand their exposition to the asset to maintain the amount of hedge for consumption. This corresponds to an increase in the slope of the blue curve when b_F, and a decrease when $b_F < 0$.

Because the financial friction theory should be considered both in emerging and developed economies, the γ, θ parameter gives an important source of connection between these worlds. Lower levels of γ, θ will imply more intense exchange rate disconnection and volatility, which is consistent with emerging markets. Here we show that it will also imply higher levels of foreign assets holdings, also consistent with emerging markets. While higher values of γ, θ may still preserve some amount of exchange rate features, this exercise shows that they are consistent with lower exchange rate volatilities and lower, or none at all, reserves, a salient feature of developed economies.

[23]Not the exchange rate perceived by financial intermediaries, but the actual exchange rate would emerge in the model solution given some value for σ_q^2.

4 Conclusion

This paper contributes to the literature on foreign reserves with a reason not much considered yet, which is exchange rates. Foreign reserves may be a useful asset even when we do not consider large risks, a sovereign default, or sudden stop events. This can occur due to the general equilibrium stochastic behavior of exchange rates, which is outlined in the empirical evidence as a negative correlation between consumption or income and exchange rates, also bringing the common intuition that in bad states of nature, or when the country goes bad, the exchange rate depreciates.

When designing a model that is capable of endogenously reproducing these exchange rate features, we show that we can account for most of the reserves observed in the empirical evidence. Such design is no easy task. There is a large literature documenting many exchange rate puzzles that appear in conventional open macro models, and several proposals of correction of such models to solve some puzzles. Using one solution that can account for many of them, we show that it also endogenously generates levels of foreign assets and exchange rate volatility consistent with emerging markets values.

In general equilibrium, the country wishes to issue debt denominated in domestic currency just to finance assets denominated in foreign currency, trading domestic debt for reserves. The reason arises from the fear of a bad state that decreases consumption while increasing exchange rates. A financial friction shock is capable of generating such a mechanism, and to specify the intuition behind we show the resulting portfolio with closed-form solutions, using both a reduced form and a micro foundation for the shock. When the financial friction becomes stronger, the demand for foreign assets increases in general equilibrium.

We highlight that the standard endowment or productivity shock can't generate this comovement and we outline the intuition for it with closed-form solutions. We maintain the shock in the model to keep the connection with more traditional models. Nevertheless, endowment or productivity is still an important source of business cycle fluctuations. We show that with only endowment shocks, the country wishes to hold non-defaultable debt instead of reserves, *e.g.* a short position on foreign assets. With the two shocks present, the country switches to a long position only when the financial friction becomes relevant in the calibration. But under the same calibration that accounts for the solution of exchange rate puzzles, the general equilibrium implies a long position on foreign assets, *e.g.* positive reserves.

A Appendix: Stationary Model with Endogenous Discount Factor

It is a property of several incomplete market economy models that the equilibrium solution for the asset position of agents is non-stationary (see **Schmitt-Grohe, Uribe (2003)**). This is the case for the microfounded and baseline models considered in this paper, as can be seen by the unit-root solution for wealth accumulation w_t in Lemma 3. Since total wealth evolves following a random walk, its unconditional variance is infinite, and any nonzero shock causes its trajectory to diverge from the steady-state. This fact may imply that the computed steady-state for foreign reserves b_F in Propositions 3 and 4 are irrelevant.

In this section, we append the microfounded model with an endogenous discount factor formulation, as in **Schmitt-Grohe, Uribe (2003)**, obtain qualitative similar results to the non-stationary model and show that the steady-state portfolio position obtained in the non-stationary model is a limiting case of economies with endogenous discount factor parameter η tending to zero. In such economies, wealth position w_t is stationary, with an autoregressive coefficient $\rho^w < 1$.

Representative consumer now maximizes

$$E_0 \sum_{j=0}^{\infty} \beta_t \frac{C_t^{1-\sigma}}{1-\sigma}$$

where endogenous intertemporal discount factor β_t allows for a stable ergodic portfolio distribution in the approximated system. It is defined recursively:

$$\beta_0 = 1,\ \beta_{t+1} = \beta_t z(C_t),\ z(C_t) = \beta C_t^{-\eta}$$

with $0 < \eta < \sigma$. Foreign households now have an analogous optimization problem, with the same discount factor β_t. Households are assumed to take each β_t as exogenous, which simplifies calculations as agents do not change their consumption choices to optimize with respect to the discount factor.[24] In this formulation, higher consumption implies a lower discount factor, which induces the agent to increase savings relative to a lower consumption scenario. As a result, agents will save more (less) given a positive (negative) shock to consumption, which ultimately causes total wealth to be stationary.

[24]This assumption can be justified by assuming that each agent in a continuum of consumers chooses their individual consumption allocation while the discount factor is determined by the aggregate consumption of the economy. Relaxing this assumption to allow agents to optimize with respect to the discount factor adds complexity to the model without changing qualitatively the results obtained.

Intertemporal maximization gives modified Euler equations for each bond for households:

$$E_t\left[\beta\frac{C_{t+1}^{-\sigma}}{C_t^{\eta-\sigma}}R_t\right]=1 \tag{50}$$

$$E_t\left[\beta\frac{C_{t+1}^{-\sigma}}{C_t^{\eta-\sigma}}R_t^*\frac{Q_{t+1}}{Q_t}e^{\psi_t}\right]=1 \tag{51}$$

$$E_t\left[\beta\frac{{C_{t+1}^*}^{-\sigma}}{C_t^{*\eta-\sigma}}R_t^*\right]=1 \tag{52}$$

$$E_t\left[\beta\frac{{C_{t+1}^*}^{-\sigma}}{C_t^{*\eta-\sigma}}\left(\frac{Q_t}{Q_{t+1}}\right)R_te^{-\psi_t}\right]=1 \tag{53}$$

The rest of the model is unchanged. It is straightforward from the linearization of the Euler equations to obtain the following Lemma, analogous to Lemma 1

Lemma 4 *Let $\xi_t \equiv b_F r_t^X$ be a zero-mean shock, and assume that both countries' endowment processes are equal, but with different innovations. Assume that the agents have an endogenous discount factor with $0<\eta<\sigma$, We can reduce the linearized model into a system of two equations and two variables:*

$$E_t q_{t+1}=\rho^w q_t-\omega_1 y_t^R-\omega_2\psi_t \tag{54}$$

$$w_t=\frac{1}{\beta}w_{t-1}+\mu q_t-\frac{\gamma}{1-2\gamma}y_t^R+\xi_t, \quad \textit{where} \tag{55}$$

$$\rho^w\equiv\frac{4(\sigma-\eta)\theta\gamma(1-\gamma)+(1-2\gamma)^2}{4\sigma\theta\gamma(1-\gamma)+(1-2\gamma)^2}<1$$

and ω_1, ω_2 and μ are the same as in Lemma 1. The solution for the exchange rate and wealth compatible with the transversality condition is:

$$q_t=\hat{\lambda}_y y_t^R+\lambda_\psi\psi_t-\frac{1-\beta\rho^w}{\mu}\left(\frac{1}{\beta}w_{t-1}+\xi_t\right),$$

$$w_t=\rho^w w_{t-1}+\left(\hat{\lambda}_y\mu-\frac{\gamma}{1-2\gamma}\right)y_t^R+\nu\psi_t+\rho^w\beta\xi_t, \quad \textit{where}$$

$$\hat{\lambda}_y\equiv\left(\frac{\beta[\sigma(1-\rho^y)-\eta](1-2\gamma)}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2}+\frac{(1-2\gamma)(1-\rho^w\beta)}{2\theta(1-\gamma)-(1-2\gamma)}\right)\frac{1}{1-\beta\rho^y}>0$$

In this modified system, wealth is now stationary so long as $\eta>0$, and this solution converges to the exogenous discount factor model when we take $\eta\to 0$. Solving for the steady-state position of reserves requires the same steps as the microfounded model, described in Appendix B.

Proposition 6 *In the incomplete markets version of the model with endogenous discount factor, the optimal (long-run) portfolio on foreign currency is*

$$b_F^{EDF} = \frac{1-2\gamma+(\sigma+\frac{\beta}{1-\beta}\eta)(1-2\theta(1-\gamma))}{\sigma(1-2\gamma)} \frac{\gamma}{1-\beta\rho^{y^R}} \frac{4\sigma\theta\gamma(1-\gamma)+(1-2\gamma)^2}{4\sigma\theta\gamma(1-\gamma)+(1-2\gamma)^2+4\frac{\beta}{1-\beta}\eta\theta\gamma(1-\gamma)}$$
$$+ \frac{(2\theta\gamma(1-\gamma)-\gamma(1-\gamma))(1-2\gamma)}{\hat{\lambda}_y\sigma(1-\beta)[4\sigma\theta\gamma(1-\gamma)+(1-2\gamma)^2+4\frac{\beta}{1-\beta}\eta\theta\gamma(1-\gamma)]}\left(\frac{\beta}{1-\beta\rho^{\psi}}\right)^2\frac{\sigma_\psi^2}{\sigma_y^2}$$

Under the calibration $\beta\rho^{\psi} \to 1$, the domestic country takes a long position on the foreign asset, $b_F^{EDF} > 0$.

B Proofs

B.1 Proof of Lemma 1

We linearize the equations that characterize the equilibrium system around the steady-state, and treat the endogenous real excess return of the foreign asset $b_F r_t^X \equiv \xi_t$ as an i.i.d. zero mean shock, since $E_t r_{t+1}^X = 0$ and only the first moment of the distribution enters the linearized system solution. By symmetry, we have that in the steady-state $Q_{ss} = P_{F,ss} = P_{H,ss} = P^*_{F,ss} = P^*_{H,ss} = 1$, $Y_{ss} = C_{ss} = Y^*_{ss} = C^*_{ss}$, $C_{H,ss} = C^*_{F,ss} = (1-\gamma)Y_{ss}$, $C_{F,ss} = C^*_{H,ss} = \gamma Y_{ss}$ and $R_{ss} = R^*_{ss} = 1/\beta$.

Before characterizing the linear equilibrium system it is important to note that in a first order approximation we cannot determine the equilibrium bond allocations, since the linear model is risk neutral and there is a continuum of bond demands that satisfies the market clearing condition while also satisfying the equilbrium equations of the system. We drop four variables $B_{H,t}, B_{F,t}, B^*_{H,t}$ and $B^*_{F,t}$ from the linear system, rewriting the system in terms of total wealth $W_t = B_{H,t} + B_{F,t}$ and $W^*_t = B^*_{H,t} + B^*_{F,t}$, and we also drop two bond market clearing conditions. The linearized equilibrium equations are:

- Terms of trade (2):

$$q_t = (1-2\gamma)s_t \tag{56}$$

- Domestic good market clearing (14):

$$y_t = (1-\gamma)c_t + \gamma c^*_t + 2\theta\gamma(1-\gamma)s_t \tag{57}$$

- Foreign good market clearing (15):

$$y_t^* = (1-\gamma)c_t^* + \gamma c_t - 2\theta\gamma(1-\gamma)s_t \tag{58}$$

- Domestic budget constraint (17):

$$w_t = \frac{1}{\beta}w_{t-1} + r_t^X b_F + \gamma\left(\frac{(2\theta(1-\gamma)-1)}{1-2\gamma}q_t - c_t + c_t^*\right) \tag{59}$$

- Euler equations (20 - 23):

$$r_t = \sigma E_t \Delta c_{t+1} \tag{60}$$
$$r_t^* = \sigma E_t \Delta c_{t+1} - E_t \Delta q_{t+1} - \psi_t \tag{61}$$
$$r_t^* = \sigma E_t \Delta c_{t+1}^* \tag{62}$$
$$r_t = \sigma E_t \Delta c_{t+1}^* + E_t \Delta q_{t+1} + \psi_t \tag{63}$$

Now, combine equations (56 - 58) to obtain

$$c_t^R + \frac{4\theta\gamma(1-\gamma)}{(1-2\gamma)^2}q_t = \frac{1}{1-2\gamma}y_t^R \tag{64}$$

which substituting back in (59), and using the assumption $b_F r_t^X \equiv \xi_t$, yields the first equation of the system:

$$w_t = \frac{1}{\beta}w_{t-1} + \mu q_t - \frac{\gamma}{1-2\gamma}y_t^R + \xi_t, \tag{65}$$

$$\text{where} \quad \mu \equiv \frac{2\gamma\theta(1-\gamma) - \gamma(1-2\gamma)}{(1-2\gamma)^2}.$$

Combining the linear Euler equations (60 - 63) we obtain the modified UIP condition:

$$r_t - r_t^* = \sigma E_t \Delta c_{t+1}^R = E_t \Delta q_{t+1} + \psi_t. \tag{66}$$

Substituting out the relative consumption from equation (64) we obtain the second equation that describes the evolution of the linear system:

$$E_t q_{t+1} = q_t - \omega_1 y_t^R - \omega_2 \psi_t \tag{67}$$

$$\text{where} \quad \omega_1 \equiv \frac{(1-2\gamma)(1-\rho^y)\sigma}{4\sigma\theta\gamma(1-\gamma) + (1-2\gamma)^2} > 0, \quad \omega_2 \equiv \frac{(1-2\gamma)^2}{4\theta\sigma\gamma(1-\gamma) + (1-2\gamma)^2} > 0.$$

Equations (65) and (67) make up a linear difference system on two endogenous variables

$E_t q_{t+1}$ and w_t, one forward-looking and one backward-looking, and three exogenous shocks y_t^R, ψ_t and ξ_t, which can be solved by using classical **BLANCHARD & KAHN (1980)** conditions. □

B.2 Proof of Proposition 1

In Lemma 1 we characterized the equilibrium system in terms of only two linear equations over $E_t q_{t+1}$ and w_t: (65) and (67). We seek an unique solution that is stable:

$$E_t \lim_{j\to\infty} \beta^j q_{t+j} = E_t \lim_{j\to\infty} \beta^j w_{t+j} = 0 \tag{68}$$

or, in other words, solutions are not expect to diverge away from the steady-state for a given random trajectory of the shocks. We start by rewriting the system in matricial form:

$$E_t \begin{bmatrix} q_{t+1} \\ w_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mu & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} q_t \\ w_{t-1} \end{bmatrix} + \begin{bmatrix} -\omega_1 & 0 & -\omega_2 \\ -\frac{\gamma}{1-2\gamma} & 1 & 0 \end{bmatrix} \begin{bmatrix} y_t^R \\ \xi_t \\ \psi_t \end{bmatrix}$$

$$\Rightarrow E_t \boldsymbol{z}_{t+1} = \boldsymbol{A}\boldsymbol{z}_t + \boldsymbol{B}\boldsymbol{u}_t$$

The eigenvalues of matrix $\boldsymbol{A}$ are 1 and $1/\beta > 1$. **BLANCHARD & KAHN (1980)** conditions for the existence of a stable solution require that we have as many unstable (e.g. greater than 1) eigenvalues as there are forward-looking variables, which is satisfied in this case. Multiply both sides by the eigenvector $\boldsymbol{v'} = [1, (1-\beta)/\beta\mu]$ of A' associated with the eigenvalue $1/\beta$ to get

$$E_t x_{t+1} = \frac{1}{\beta} x_t + \boldsymbol{B'}\boldsymbol{u}_t, \quad x_t \equiv \boldsymbol{v'}\boldsymbol{z}_t, \boldsymbol{B'} \equiv \boldsymbol{v'}\boldsymbol{B}$$

Iterating the equation forward and imposing the transversality condition (68), we obtain

$$x_t = -\beta E_t \sum_{j=0}^{\infty} \beta^j \left[\frac{1-\beta}{\mu\beta} \xi_{t+j} - \left(\omega_1 + \frac{\gamma(1-\beta)}{(1-2\gamma)\beta\mu} \right) y_{t+j}^R - \omega_2 \psi_{t+j} \right]$$

We can use the definition of the AR(1) processes y_t^R and ψ_t and the definition of x_t to obtain the equation:

$$q_t + \frac{1-\beta}{\beta\mu} w_{t-1} = -\frac{1-\beta}{\mu} \xi_t + \left(\beta\omega_1 + \frac{\gamma(1-\beta)}{(1-2\gamma)\mu} \right) \frac{1}{1-\beta\rho^y} y_t^R + \frac{\beta\omega_2}{1-\beta\rho^\psi} \psi_t$$

which, by aggregating parameters and a bit of rearranging yields the desired result:

$$q_t = \lambda_y y_t^R + \lambda_\psi \psi_t - \frac{1-\beta}{\mu}\left(\frac{1}{\beta}w_{t-1} + \xi_t\right), \quad \text{where}$$

$$\lambda_y \equiv \left(\frac{\beta\sigma(1-2\gamma)(1-\rho^y)}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2} + \frac{(1-2\gamma)(1-\beta)}{2\theta(1-\gamma)-(1-2\gamma)}\right)\frac{1}{1-\beta\rho^y} > 0$$

$$\lambda_\psi \equiv \frac{(1-2\gamma)^2}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2}\frac{\beta}{1-\beta\rho^\psi} > 0$$

We also obtain the process for w_t applying back the result on real exchange rate q_t on equation (65):

$$w_t = w_{t-1} + \left(\lambda_y\mu - \frac{\gamma}{1-2\gamma}\right) y_t^R + \nu\psi_t + \beta\xi_t \tag{69}$$

$$\text{where} \quad \nu \equiv \left(\frac{\beta}{1-\beta\rho^\psi}\right)\frac{2\gamma\theta(1-\gamma)-\gamma(1-2\gamma)}{4\theta\sigma\gamma(1-\gamma)+(1-2\gamma)^2}$$

□

B.3 Proof of Corollary 1

In Lemma 1 we have shown the solution of the linear system for relative consumption c_t^R, in equation (64), which depends only on current real exchange rate q_t and the exogenous relative endowment shock y_t^R. Substituting our solution for q_t, (29), and rearranging parameters we immediatelly obtain equation (1). □

B.4 Proof of Lemma 2

From the previous solution for the real exchange rate process q_t and relative consumption c_t^R, we can obtain the process for the ex-post excess return on the foreign asset, given an external position on reserves b_F, by calculating $r_{t+1}^X \equiv q_{t+1} - q_t + r_t^* - r_t + \psi_t$, and using the fact derived from equation (66) that $r_t^* - r_t = \sigma E_t \Delta c_{t+1}^R$. First, we obtain the ex-post real exchange rate:

$$q_{t+1} = q_t - \omega_1 y_t^R - \omega_2\psi_t + \lambda_y \xi_{y+1}^{y^R} + \lambda_\psi \xi_{t+1}^\psi - \frac{1-\beta}{\mu}\xi_{t+1} \tag{70}$$

Ex-post relative consumption change is, by substituting the terms on equation (1) and aggregating parameters:

$$c_{t+1}^R = c_t^R - \frac{\omega_1}{\sigma}y_t^R + \frac{1-\omega_2}{\sigma}\psi_t + \Theta_1\xi_{t+1}^{y^R} + \Theta_2\xi_{t+1} - \Theta_3\xi_{t+1}^\psi \tag{71}$$

which implies ex-post differential return as

$$r_t - r_t^* = \sigma E_t \Delta c_{t+1}^R = \omega_1 y_t^R + (1-\omega_2)\psi_t \tag{72}$$

Equations (70) and (72) gives total excess return:

$$r_{t+1}^X \equiv q_{t+1} - q_t + r_t^* - r_t + \psi_t = \lambda_y \xi_{y+1}^{y^R} + \lambda_\psi \xi_{t+1}^\psi - \frac{1-\beta}{\mu}\xi_{t+1} \tag{73}$$

By definition, $\xi_{t+1} = b_F r_{t+1}^X$ and we can isolate equation (73) in terms of r_{t+1}^X to obtain the desired result. □

B.5 Proofs of Propositions 2 and 3

As in **DEVEREUX and SUTHERLAND (2014)**, we use a second-order approximation of the Euler equations of the system (20-23) to obtain a condition that: i) determines the zero-order portfolio allocation; and ii) depends only on first-order terms. The exact condition we obtain for the baseline model is equation (24), and by substituting the previous solution for the real exchange rate and relative consumption we obtain:

$$\begin{aligned} E_t\Bigg[\Bigg(&\frac{1-\beta}{1-\beta\rho^y}\frac{\sigma(2\theta(1-\gamma)-1)-(1-2\gamma)}{2\theta(1-\gamma)-(1-2\gamma)}\xi_{t+1}^{y^R} - \frac{\beta}{1-\beta\rho^\psi}\xi_{t+1}^\psi \\ &+ (1-\beta)\frac{4\sigma\theta\gamma(1-\gamma)+(1-2\gamma)^2}{2\theta\gamma(1-\gamma)-\gamma(1-2\gamma)}\xi_{t+1}\Bigg)r_{t+1}^X\Bigg] = 0 \end{aligned} \tag{74}$$

Substituting the definition of the excess return it is straightforward to establish:

$$E_t \xi_{t+1}^{y^R} r_{t+1}^X = \frac{\lambda_y \sigma_y^2}{1+\frac{1-\beta}{\mu}b_F} \tag{75}$$

$$E_t \xi_{t+1}^{\psi} r_{t+1}^X = \frac{\lambda_\psi \sigma_\psi^2}{1+\frac{1-\beta}{\mu}b_F} \tag{76}$$

$$E_t \xi_{t+1} r_{t+1}^X = \frac{b_F}{1+\frac{1-\beta}{\mu}b_F}\left[\frac{(\lambda_y)^2\sigma_y^2}{1+\frac{1-\beta}{\mu}b_F} + \frac{(\lambda_\psi)^2\sigma_\psi^2}{1+\frac{1-\beta}{\mu}b_F}\right] \tag{77}$$

After substituting back in equation (74) and simplifying terms we obtain a solution of b_F in the form

$$b_F = \frac{1 - 2\gamma + \sigma(1 - 2\theta(1-\gamma))}{\sigma(1-2\gamma)} \frac{\gamma}{1 - \beta\rho^y} + \Omega \left(\frac{\beta}{1 - \beta\rho^\psi} \right)^2 \frac{\sigma_\psi^2}{\sigma_y^2} \tag{78}$$

$$\text{where} \quad \Omega \equiv \frac{(2\theta\gamma(1-\gamma) - \gamma(1-2\gamma))(1-2\gamma)}{\sigma\lambda_y(1-\beta)(4\sigma\theta\gamma(1-\gamma) + (1-2\gamma)^2)} > 0$$

Setting $\psi_t = 0$ is the same as setting $\sigma_\psi = 0$, which gives the position on the foreign asset when there is no risk-premium shock in the model:

$$b_F^0 = \frac{1 - 2\gamma + \sigma(1 - 2\theta(1-\gamma))}{\sigma(1-2\gamma)} \frac{\gamma}{1 - \beta\rho^y}$$

To see that $b_F^0 < 0$, all we need to check is that the numerator of the first fraction in (32) is negative. But it is easy to see that this is equivalent to

$$\gamma < \frac{1}{2} + \frac{\sigma(\theta - 1)}{2(\sigma\theta - 1)}$$

which is true for every $\gamma < 1/2$ since σ, $\theta > 1$.

Since $\Omega > 0$, it is clear that taking $\beta\rho^\psi \to 1$ in equation (78) implies that $b_F > 0$, which completes the proof. □