A Spatial Extension of Synthetic Difference-in-Differences

Renan Serenini * Frantisek Masek †

Sapienza University of Rome

Abstract

We propose a spatial extension of the Synthetic Difference-in-Differences (SDiD) estimator of Arkhangelsky et al. (2021). Our estimator handles the situation of a possible violation of the Stable Unit Treatment Value Assumption (SUTVA) when treatment may spillover to control units included in the donor pool resulting in biased and inconsistent Average Treatment Effect (ATE) estimation. We build on the approach of the Spatial Difference-in-Differences estimator of Delgado and Florax (2015) and incorporate it into SDiD. Thus, the ATE can be disentangled into direct and indirect treatment effects. We compare empirically our approach with both estimators in situations where they would be commonly used. We show that it can be superior to Delgado and Florax (2015) for estimating the indirect effect, while still keeps the advantages of Arkhangelsky et al. (2021) to estimate the direct effect.

Keywords: Average Treatment Effect, Stable Unit Treatment Value Assumption, Treatment Spillovers, Synthetic Control, Difference-in-Differences, Spatial Econometrics

JEL classification: C21, C23, D62, I18

*renan.serenini@uniroma1.it.

†frantisek.masek@uniroma1.it.

We gratefully thank the Center for Spatial Data Science (CSDS) at the University of Chicago for hosting us during the time of writing the article. Namely, in particular, we thank Julia Koschinsky and Luc Anselin for their support and enriching discussions and to all the participants of the CSDS study group for their valuable comments. We are also grateful for the helpful feedbacks of Dmitry Arkhangelsky and the other participants of the 2023 American Causal Inference Conference.

1 Introduction

The Synthetic Difference in Differences (SDiD) estimator of Arkhangelsky et al. (2021) offers an intriguing merge of ideas coming from two widely used tools in the quasi-experimental literature - Synthetic Control Method(s) (SCM) and Difference-in-Differences (DiD). The authors show that using reweighting of control units delivered by the synthetic control component within a two-way fixed setting may be competitive, or possibly superior, in situations where each of the aforementioned approaches would be deployed individually. While the reweighting weakens the reliance on the parallel trend assumptions, unlike in the conventional SCM, SDiD is also invariant to additive unit-level shifts as it is common in the DiD literature.[1]

To the best of our knowledge, there has not been developed any extension of Arkhangelsky et al. (2021) that would be able to tackle the situation of possible spillovers of the treatment. In such a situation, the Stable Unit Treatment Value Assumption (SUTVA) is violated and standard quasi-experimental methods will be biased and inconsistent. In DiD, some approaches try to handle this problem by extending the DiD estimators for spatial components (Delgado and Florax, 2015; Butts, 2021*a*). The case when treatment may spill over to other units is studied also in the SCM literature (Cao and Dowd, 2019; Di Stefano and Mellace, 2020). Yet no such counterpart has appeared for the SDiD estimator.

We come with a spatial extension of SDiD building on the approach of Delgado and Florax (2015). In their DiD for spatial data, treatment can spill over to other units otherwise considered in the control group. Hence, the potential outcome of observed units depends not only on their own treatment status, but also on treatment of units experiencing spillovers. Delgado and Florax (2015) control for this process by utilizing spatial econometrics tools as the spatial weights matrix to incorporate treatment spillovers (Anselin, 1988). Specifically, the spatial interaction related to treatment responses makes their approach isomorphic to the SLX model (Vega and Elhorst, 2015). In our approach, we incorporate Delgado and Florax (2015) into Arkhangelsky et al. (2021) to develop a spatial SDiD (SSDiD).

Using two crafted simulation studies, we demonstrate that the spatial treatment extension of the SDiD substantially improves the estimation of the Average Treatment Effect (ATE)

[1]For the sake of clarity, we do not elaborate on other features that make the estimator of Arkhangelsky et al. (2021) appealing as weighting of time periods or using regularization. Although these may improve the properties of the estimator, they are not at the core of our interest. However, they may be used in our spatial extension of the SDiD too.

in the situation of the SUTVA violation. Our estimator can capture both direct and indirect treatment effects, as it is the case for the conventional DiD in the spatial extension of Delgado and Florax (2015). What is more, we do not alter in any way the comparison of the direct treatment effect between SDiD and DiD. Therefore, the characteristics related to the comparison discussed in Arkhangelsky et al. (2021) hold also in our approach.

A more nuanced discussion is necessary when comparing our estimator to Delgado and Florax (2015) for the case of the indirect treatment effect. Given that we use the same weights stemming from the synthetic control constructed for directly treated units also for the estimation of the indirect treatment effect, we assume a close similarity of directly and indirectly treated units. If this assumption holds and hence the synthetic control constructed for directly treated units is better than the uniform weighting embedded in the spatial DiD, our estimator improves on the approach of Delgado and Florax (2015) also in the estimation of the indirect treatment effect. In the simulated studies, we show that this is usually the case.

The units that received the treatment spillover because of interactions with directly treated units must be similar enough to treated units. In such a situation, the weights obtained for directly treated units cause that also indirectly treated units are compared to more similar units rather than to all the units in the donor pool. The more similar units are of a greater importance given that these are similar for both directly and indirectly treated units, unlike in Delgado and Florax (2015) where each unit in the donor pool obtains the same weight.

Related literature. The closest to our estimator is the work of Delgado and Florax (2015). Their spatial DiD can be even seen as a special case of our approach in the situation of uniform weights. Using Monte Carlo simulations, they show that incorporating spatial interaction of treatment decreases bias caused by the SUTVA violation when comparing their estimator to a simple DiD setup. Hence, they can estimate both direct and indirect treatment effects. Their method can be perceived as the SLX model coming from the spatial econometrics literature (Vega and Elhorst, 2015). As already mentioned above, we follow up on Delgado and Florax (2015) and extend their method using reweighting of the units in the donor pool.

A different perspective is taken in the work of Butts (2021*a*) who discusses two sources of the bias due to the presence of spillovers.[2] The author mentions the fact that the control units affected by the treatment spillovers cannot serve as counterfactual trend, given that their

[2] Although Butts (2021*a*) grasps the problem from a distinct angle, one can show that his approach is isomorphic to the estimator of Delgado and Florax (2015).

outcomes are affected by treatment. On top of that, he points out that also changes in the treated units' outcomes may be influenced by the treatment effects of the close units through sort of general equilibrium forces. He suggests modeling spillovers in a general form using 'Rings' style estimator. Specifically, Butts (2021*a*) proposes the way of handling both sources of bias while semi-parametrically estimating possible spillovers. Moreover, he embeds the staggered treatment into his method. In his following work, Butts (2021*b*) presents data-driven ring selection process. The estimator then compares units immediately next to treatment (an inner-ring) to units just slightly further away (an outer-ring).

Another pioneer in the DiD spillovers-broadened literature is Clarke (2017). His estimator takes into account treatment spillovers using two classes of estimands - besides standard treatment effects, it works also with so-called "close" to treatment effects. The method offers a procedure of defining the distance through which treatment propagates, while distance may be defined even as a multidimensional measure.

Conceptually distinct, DiD estimator is used in Dubé et al. (2014) to estimate the effect of public mass transit systems on real-estate prices. The authors conduct the DiD analysis within the Spatial Autoregressive model (SAR) (Anselin, 1988). Hence, given the global property of the SAR model, their approach allows for general equilibrium feedback effects that one cannot obtain using the method of Delgado and Florax (2015). Kolak and Anselin (2020) discuss their approach in a thorough survey of various attempts extending quasi-experimental methods used in the causal inference literature by spatial aspects. Their article may serve as a coherent summary of many approaches that are developed to handle the SUTVA violation.

Moving to the Synthetic Control Method(s) literature, we stress two articles that consider the spillovers problem. Di Stefano and Mellace (2020) develop the Inclusive Synthetic Control Method which allows including even indirectly affected units in the donor pool. Given that their method does not need to restrict the donor pool into pure controls and affected units, it can be useful in applications where incorporating even the indirectly treated units is unavoidable to get a reasonable control unit. Di Stefano and Mellace (2020) use the case of the German reunification to show the method's implementation and its comparison to the conventional Synthetic Control Method of Abadie et al. (2015) and the restricted version of SCM from Abadie and L'Hour (2021).

Cao and Dowd (2019) propose method-wise different way of estimating both direct treatment effects and spillover effects. Cao and Dowd (2019) impose a linear assumption on both direct treatment effect and spillover effects. They show that their estimations are asymptotically unbiased, while building even an inferential procedure that is asymptotically unbiased too. The method can be used in situations with multiple treated units or periods, assuming that the underlying factor model is either stationary or cointegrated.

2 The Spatial Synthetic Difference-in-Differences foundation

Synthetic Difference-in-Differences. Arkhangelsky et al. (2021) show that their estimator may be written as the following optimization problem:

$$\left(\hat{\tau^{sdid}}, \hat{\mu}, \hat{\alpha}, \hat{\beta}\right) = \arg\min_{\mu,\alpha,\beta,\tau} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} \left[Y_{it} - \left(\mu + \alpha_i + \beta_t + \tau D_{it}\right) \right]^2 \hat{\omega}_i^{sdid} \hat{\lambda}_t^{sdid} \right\}, \quad (1)$$

where we have a two-way fixed effects regression multiplied by nonuniform unit weights $\hat{\omega}_i^{sdid}$ and time weights $\hat{\lambda}_t^{sdid}$. The former comes from the synthetic control algorithm which is altered compared to the conventional SCM of Abadie and Gardeazabal (2003) and Abadie et al. (2010). On top of that, Arkhangelsky et al. (2021) also introduce time weighting in order to match the average post-treatment outcome with the pre-treatment outcomes for each control unit. The rest of the notation follows Arkhangelsky et al. (2021) with one exception. The treatment status is denoted as D because W used in Arkhangelsky et al. (2021) will be later used for the spatial weights matrix.

We do not change anything in the way of obtaining $\hat{\omega}_i^{sdid}$ and $\hat{\lambda}_t^{sdid}$. The algorithms to get the both type of weights fully follow Arkhangelsky et al. (2021). Hence, in the case of $\hat{\omega}_i^{sdid}$ we have:

$$(\hat{\omega_0}, \hat{\omega}_i^{sdid}) = \arg\min_{\omega_0 \in \mathbb{R}, \omega \in \Omega} \ell_{unit}(\omega_0, \omega), \quad (2)$$

where

$$\ell_{unit}(\omega_0, \omega) = \sum_{t=1}^{T_{pre}} \left(\omega_0 + \sum_{i=1}^{N_{co}} \omega_i Y_{it} - \frac{1}{N_{tr}} \sum_{i=N_{co}+1}^{N} Y_{it} \right)^2 + \zeta^2 T_{pre} ||\omega||_2^2, \quad (3)$$

$$\Omega = \left\{ \omega \in \mathbb{R}_+^N : \sum_{i=1}^{N_{co}} \omega_i = 1, \quad \omega_i = N_{tr}^{-1} \ \forall i = N_{co}+1, ..., N \right\}, \quad (4)$$

with $\mathbb{R}_+^N$ denoting the positive real line while the regularization parameter is set in accordance with Arkhangelsky et al. (2021). The problem described in 2, 6, and 4 deviates from the conventional SCM algorithm from Abadie and Gardeazabal (2003), Abadie et al. (2010) or Abadie et al. (2015) in two aspects. Firstly, including the intercept ω_0 means that we want to fit trends instead of levels. Second, the regularization parameter makes the synthetic unit less sparse; i.e., the weights are more dispersed and hence the synthetic control does not rely on large weights of only few units. What is more, the penalization also helps to ensure the uniqueness of the weights.

The isomorphic minimization problem is set up to find the time weights. The only difference is the absence of the regularization parameter. Thus, the problem follows:

$$(\hat{\lambda_0}, \hat{\lambda}_i^{sdid}) = \arg\min_{\lambda_0 \in \mathbb{R}, \lambda \in \Lambda} \ell_{time}(\lambda_0, \lambda), \tag{5}$$

where

$$\ell_{unit}(\lambda_0, \lambda) = \sum_{i=1}^{N_{co}} \left(\lambda_0 + \sum_{t=1}^{T_{pre}} \lambda_t Y_{it} - \frac{1}{T_{post}} \sum_{t=T_{pre}+1}^{T} Y_{it} \right)^2, \tag{6}$$

$$\Lambda = \left\{ \lambda \in \mathbb{R}_+^T : \sum_{t=1}^{T_{pre}} \lambda_t = 1, \quad \lambda_t = T_{post}^{-1} \ \forall t = T_{pre}+1, ..., T \right\}, \tag{7}$$

while the idea behind using the time weights is to match the average post-treatment outcomes for control units (up to a constant) by reweighting pre-treatment periods. Therefore, when estimating the ATE within a regression, only a subset of pre-treatment periods is taken into account.

Spatial Synthetic Difference-in-Differences. In order to extend SDiD by a spatial component of the treatment spillover, we take a step back and start from the two-way fixed effects estimator which can be expressed as:

$$\left(\hat{\tau^{did}}, \hat{\mu}, \hat{\alpha}, \hat{\beta} \right) = \arg\min_{\mu,\alpha,\beta,\tau} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} \left[Y_{it} - (\mu + \alpha_i + \beta_t + \tau D_{it}) \right]^2 \right\} \tag{8}$$

One can add the spatial extension following Delgado and Florax (2015). Thus, we have:

$$\left(\hat{\tau^{spatialdid}}, \hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\tau}, \hat{\rho} \right) = \arg\min_{\mu,\alpha,\beta,\tau,\rho} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} \left[Y_{it} - \left(\mu + \alpha_i + \beta_t + \tau (I + \rho W) D_{it} \right) \right]^2 \right\}, \tag{9}$$

where W is a $TN \times TN$ block-diagonal row-standardized spatial weights matrix that contains non-zero elements for spatial units within a given neighborhood criterion while ρ stands for a spatial autoregressive parameter driving the strength of the spatial interaction in treatment.[3] In other words, equation 9 is the regression from Delgado and Florax (2015) reshuffled as an optimization ordinary least square problem. One can write $\tau^s = \rho\tau$ and reformulate the expression as follows:

$$\left(\hat{\tau}, \hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\tau}_s \right) = \arg\min_{\mu,\alpha,\beta,\tau,\tau_s} \sum_{i=1}^{N} \sum_{t=1}^{T} \left[Y_{it} - (\mu + \alpha_i + \beta_t + \tau D_{it} + W \tau^s D_{it}) \right]^2, \tag{10}$$

[3]Naturally, T stands for the total number of time periods while N denotes the number of units.

where both τ and τ_s must be estimated. While the former measures a direct treatment effect, the latter an indirect effect. Naturally, if we merge 1 and 10, we yield:

$$\left(\hat{\tau}, \hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\tau}_s\right) = \arg\min_{\mu,\alpha,\beta,\tau,\tau_s} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} \left[Y_{it} - \left(\mu + \alpha_i + \beta_t + \tau D_{it} + W\tau^s D_{it}\right) \right]^2 \hat{\omega}_i \hat{\lambda}_t \right\} \quad (11)$$

Note that the presence of the treatment spillover alters the conditional ATE definition. To simplify the notation, consider $t = 1$ being the post-treatment period and $t = 0$ the pre-treatment period.[4] Given that now all units (including directly treated) can be indirectly treated through spillovers coming from the treated units ($0 < wd < 1$ and $wd \in WD$), we have:

$$\begin{aligned} ATE(wd) = \{&\mathbb{E}[Y|D = 1, t = 1, WD = wd] - \mathbb{E}[Y|D = 1, t = 0, WD = wd]\} \\ -\{&\mathbb{E}[Y|D = 0, t = 1, WD = 0] - \mathbb{E}[Y|D = 0, t = 0, WD = 0]\}, \end{aligned} \quad (12)$$

which gives us:

$$ATE(wd) = \tau + \tau^s wd = \tau(1 + \rho wd), \quad (13)$$

where following Delgado and Florax (2015) and working with $\overline{WD}$ as the average proportion of treated neighbors enables us to rewrite it as follows:

$$ATE = \mathbb{E}[ATE(wd)|WD] = \tau(1 + \rho\overline{WD}) \quad (14)$$

Expression 14 shows that the ATE will be biased in case we omit the spillover effects of the treatment. Moreover, we can disentangle the ATE into Average Treatment on the Treated (ATT) and Average Indirect Treatment Effect (AITE). In the case of the former, we get:

$$\begin{aligned} ATT = \{&\mathbb{E}[Y|D = 1, t = 1, WD = 0] - \mathbb{E}[Y|D = 1, t = 0, WD = 0]\} \\ -\{&\mathbb{E}[Y|D = 0, t = 1, WD = 0] - \mathbb{E}[Y|D = 0, t = 0, WD = 0]\}, \end{aligned} \quad (15)$$

which is equivalent to the standard ATE in the case when the SUTVA holds. The IATE can be written as:

$$\begin{aligned} AITE(wd) = \{&\mathbb{E}[Y|D = 0, t = 1, WD = wd] - \mathbb{E}[Y|D = 0, t = 0, WD = wd]\} \\ -\{&\mathbb{E}[Y|D = 0, t = 1, WD = 0] - \mathbb{E}[Y|D = 0, t = 0, WD = 0]\} \end{aligned} \quad (16)$$

Using the notation from 11, we can write $ATT = \tau$ and $AITE(wd) = \tau^s wd$. The later equation can be again reshuffled into $AITE = \tau^s\overline{WD}$. Thus, we can estimate both direct and indirect treatment effects using SSDiD.

To provide a concise step-by-step summary of what the SSDiD estimator is about and how it is generated, we write down a detailed description of all necessary steps. The estimator in 11 can be attained by obeying the following algorithm:

[4]Hence, think in terms of the simplest two-periods textbook DiD case.

Algorithm 1 Spatial Synthetic Difference in Differences

Input: Data for Y and D. Consider N units in total of which N_{tr} are treated directly and N_{sp} indirectly.

Output: $\hat{\tau}$ and $\hat{\tau^s}$

1: Construct the spatial weight matrix W.

2: Separate the donor pool into units receiving spillover (N_{sp}) of treatment and those not receiving any spillover (N_{co}). Let us call the units in the latter group pure controls.

3: Compute regularization parameter ζ following Arkhangelsky et al. (2021).

4: Obtain $\hat{\omega}_i$ and $\hat{\lambda}_t$ stemming from the approach of Arkhangelsky et al. (2021) using only pure control units in the donor pool (N_{co}).

5: For indirectly treated units, use $\hat{\omega}_i = N_{sp}^{-1}$. Substitute $\hat{\omega}_i = N_{tr}^{-1}$ in the case of directly treated units.

6: Compute the SSDiD estimator by running the following weighted regression:

$$\left(\hat{\tau}, \hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\tau}, \hat{\tau_s}\right) = \arg\min_{\mu,\alpha,\beta,\tau,\tau_s} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} \left[Y_{it} - \left(\mu + \alpha_i + \beta_t + \tau D_{it} + W\tau^s D_{it}\right)\right]^2 \hat{\omega}_i \hat{\lambda}_t \right\}$$

3 Controlled simulations

The goal of this section is to demonstrate that our estimator is suitable for situations of the SUTVA violation. Specifically, we consider the situation where treatment affects also some of the units included among the control group. If we ignore the treatment spillovers, we end up with a biased and inconsistent estimator of the ATE. If the estimand of interest is only the Average Treatment on the Treated (ATT), we can exclude the indirectly affected units, as commonly recommended by the literature, as apply the SDiD, for instance. However, in this way neither the AITE nor the ATE can be estimated. The approach of Delgado and Florax (2015) allows doing this, but we demonstrate we can be more efficient as we exploit of the advantages of the SDiD over the traditional DiD. The idea of Delgado and Florax (2015) is to take the spatial component of treatment into account within the conventional DiD setting. In our case, we offer an extension for the case of the SDiD where we reweight the units in the control group by its proximity towards the directly treated units.

We show that our approach can recover the ATT even after manipulating the donor pool and including back units that are not considered in the estimation process of Arkhangelsky et al. (2021). On top of that, we show that the estimator can also estimate an unbiased Average

Indirect Treatment Effect, which is the main advantage of our approach. We study different cases of the treatment spillover strength by varying the parameterization of ρ.

In the first round of simulations we use data at county level, with multiple treated units, a setting closer to DiD studies. Four states are used as example. The number of counties in each state is written inside the parenthesis in Table 1. We use series for monthly unemployment rate for the time period of 2002 to 2004, then 36 observations for each county. The treatment occurs at the end of the second year, so we have 24 pre-treatment periods and 12 post-treatment periods. In each iteration, a random number of counties (from 5 to 10% of the total counties) is treated, and the effect spills over to the immediate neighbors (first-order queen contiguity matrix) of the treated counties. Figure 1 show a map of a single iteration for the state of Utah:

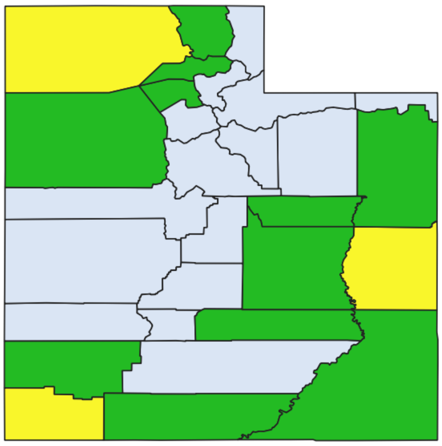

Figure 1: Counties in yellow are treated and counties in green receive the spillover effect.

For every iteration, we take real data and replace the post-treatment data summing up the ATT (and the respective spillover effect) to the original series. The ATT is chosen as 25% of the average unemployment rate of the entire period for all units. We point out that although we are imposing a simulated effect, the estimation process to obtain the unit weights in the SDiD considers just the pre-treatment period, so only real data is used to fit the synthetic series. For every value of ρ and every state, ten thousand simulations were performed. Table 1 shows the results for the estimation of the ATT:

Table 1: Bias of the ATT by the Spatial SDiD

ρ	**Arizona(15)**	**Utah(29)**	**Dakota(53)**	**Illinois(102)**
0.0	-0.0285	0.0388	0.0042	-0.0089
0.25	-0.0195	0.0365	0.0033	-0.0067
0.5	-0.0402	0.029	-0.0014	-0.0075
0.75	-0.0354	0.034	-0.0091	-0.0087
0.9	-0.0224	0.0467	-0.0048	-0.0044

In this case, treated units are also allowed to receive the indirect effect, if they have treated neighbors. We consider the effects as being additive. From the results, we can see that our estimator is able to recover the ATT which a negligible bias, especially for situations with more than 50 units. For every simulation, besides the ATT we also estimate the AITE, which is our main interest. The results are shown in Table 2:

Table 2: Bias of the Average Indirect Treatment Effect

ρ	**Arizona(15)**	**Utah(29)**	**Dakota(53)**	**Illinois(102)**
0.0	-0.0454	0.0756	0.0203	-0.0188
0.25	-0.0376	0.0824	0.0186	-0.0144
0.5	-0.0369	0.0836	0.0138	-0.015
0.75	-0.0309	0.0803	0.0149	-0.0128
0.9	-0.0339	0.0815	0.0138	-0.0132

Again, the estimator is able to capture the Average Indirect Treatment Effect and for a situation with more than 50 units the bias is very tiny. interestingly, the results hold regardless of the strength of the spatial autocorrelation parameter. Summarizing, the estimator is able to retrieve both components of the ATE, the direct and the indirect effect.

In the second round of simulations we use data at State level, with just one treated unit at a time, in a setting closer to Synthetic Control Method studies. Simulations are similar to the previous one, 24 pre-treatment periods and 12 post, using unemployment rate data. In this case we do not draw a random state to be treated but instead iterate across all possible combinations, one state treated by time. Then also iterate across the time, changing the time period window from the start of 1976 to 2015. So 49 states (Hawaii excluded) times 40 years, leads to 1960 combinations/estimations. For this case ρ is fixed as 0.8 and we estimate the ATT all 3 methods and the AITE with both spatial methods, Spatial Diff-in-Diff from **?** and our Spatial Synthetic Diff-in-diff. Table 4 show the results for the estimation of the ATT:

Table 3: Descriptive Statistics for the bias of ATT

	Synth-DiD	**Spatial Synth-DiD**	**Spatial DiD**
count	1960	1960	1960
std	0.3617	0.3616	0.7126
min	-1.7453	-1.7423	-4.0227
25%	-0.2115	-0.2123	-0.4006
50%	-0.0013	0.0005	0.0227
75%	0.2201	0.2192	0.4198
max	1.9013	1.8934	3.2519

In the previous simulations, the ATT estimated by the Synthetic DiD could not be exactly compared to our approach, as treated units that also receive spillover would be removed from the analysis. Now, with a perfectly comparable scenario (just one treated unit at a time), we see that the estimations of ATT is very similar for both and quite superior to the spatial DiD of Delgado and Florax (2015). Figure 2 shows density plots with the distributions of the bias for all methods:

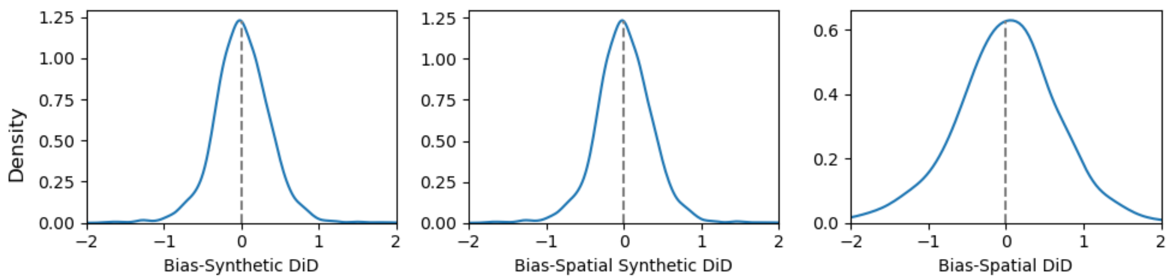

Figure 2: Density plots for the bias of ATT

It is clear that the Spatial DiD is much less precise than the others. The results are similar when analyzing the estimation of the AITE. Table 4 shows the descriptive statistics for the bias in the simulations and Figure 3 the density plots:

Table 4: Descriptive Statistics for the bias of AITE

	Spatial Synth DiD	**Spatial DiD**
count	1960.0	1960.0
std	0.2312	0.4732
min	-0.9726	-2.3528
25%	-0.1296	-0.2359
50%	0.0019	0.0076
75%	0.1398	0.2659
max	0.8991	1.7272

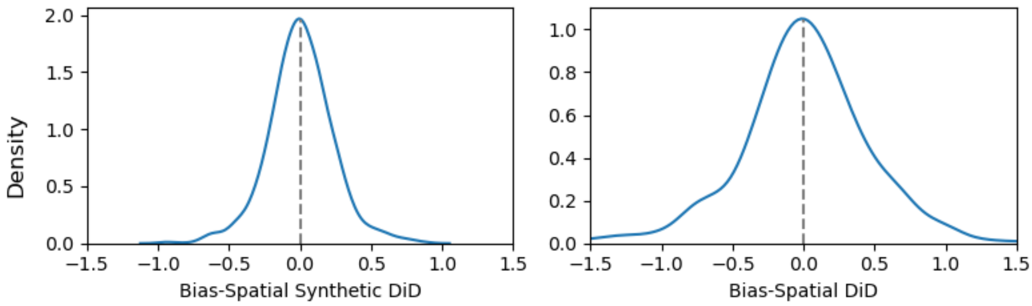

Figure 3: Caption

The Spatial Synthetic Diff-in-diff shows to be more efficient than the standard approach. When comparing our estimator to Delgado and Florax (2015), it is straightforward to see their approach as a special case of ours. Particularly, using uniform weights for units and time periods means that 11 boils down back to 10. It is thus natural to conclude that all the features presented in Arkhangelsky et al. (2021) related to the comparison of Synthetic Difference-in-Differences and conventional Difference-in-Differences carry forward when analyzing the direct treatment effect by our estimator or Delgado and Florax (2015).

From the results of the AITE it is evident that even though we use the synthetic series created for main treated unit as a control for the indirect treated units, the results are better than simply using uniform weights. This fact supports our assumption that closer units, that receive the spillover effect, are similar enough to the treated unit to be compared to the same synthetic series.

4 Conclusion

We offer an extension of the Synthetic Difference-in-Differences estimator of Arkhangelsky et al. (2021) for situations of treatment spillover. In such a case, the Stable Unit Treatment Value Assumption is violated; hence, the Average Treatment Effect estimation will be biased if we use standard Synthetic Difference-in-Differences. We exploit the spatial version of Difference-in-Differences of Delgado and Florax (2015) and include their structure that builds on the spatial weights matrix from the spatial econometrics literature (Anselin, 1988) into Synthetic Difference-in-Differences. Hence, the resulting regression equation may be summarized as the SLX model (Vega and Elhorst, 2015) estimated by a weighted least square given the reweighting of control units and pre-treatment periods coming from Arkhangelsky et al. (2021). Following this strategy, we can disentangle the Average Treatment Effect into direct and indirect effects.

Using controlled simulations, we show that our estimator can handle the situation of the SUTVA violation and get rid of bias caused by omitted variable in the form of the spatial treatment spillover. When comparing to Arkhangelsky et al. (2021) we improve in allowing the possibility of estimating the second component of the ATE, which we call AITE, impossible in their approach, as they tackle just the ATT. Thus, we get a better approximation of the ATE. Comparing to Delgado and Florax (2015) we show that exploiting the features of the Synthetic Diff-in-diff bring more precision to the estimation of both ATT and AITE, so we have a more precise estimation of both.

Bibliography

Abadie, A., Diamond, A. and Hainmueller, J. (2010), 'Synthetic control methods for comparative case studies: Estimating the effect of california's tobacco control program', *Journal of the American Statistical Association* **105(490)**, 493–505.

Abadie, A., Diamond, A. and Hainmueller, J. (2015), 'Comparative politics and the synthetic control method', *American Journal of Political Science* **59(2)**, 495–510.

Abadie, A. and Gardeazabal, J. (2003), 'The economic costs of conflict: A case study of the basque country', *American Economic Review* **93(1)**, 113–132.

Abadie, A. and L'Hour, J. (2021), 'A penalized synthetic control estimator for disaggregated data', *Journal of the American Statistical Association* **116(536)**, 1817–1834.

Anselin, L. (1988), 'Spatial econometrics: Methods and models', *Kluwer Academic, Dordrecht* .

Arkhangelsky, D., Athey, S., Hirshber, D. A., Imbens, G. W. and Wager, S. (2021), 'Synthetic difference-in-differences', *American Economic Review* **111(12)**, 4088–4118.

Butts, K. (2021*a*), 'Difference-in-differences estimation with spatial spillovers', *Papers 2105.03737, arXiv.org* .

Butts, K. (2021*b*), 'Difference-in-differences with geocoded microdata', *Papers 2110.10192, arXiv.org* .

Cao, J. and Dowd, C. (2019), 'Estimation and inference for synthetic control methods with spillover effects', *Papers 1902.07343, arXiv.org* .

Clarke, D. C. (2017), 'Estimating difference-in-differences in the presence of spillovers', *MPRA Paper 81604, University Library of Munich, Germany* .

Delgado, M. S. and Florax, R. J. (2015), 'Difference-in-differences techniques for spatial data: Local autocorrelation and spatial interaction', *Economics Letters* **137**, 123–126.

Di Stefano, R. and Mellace, G. (2020), 'The inclusive synthetic control method', *Discussion Papers on Business and Economics, University of Southern Denmark* .

Dubé, J., Legros, D., Thériault, M. and Des Rosiers, F. (2014), 'A spatial difference-in-differences estimator to evaluate the effect of change in public mass transit systems on house prices', *Transportation Research Part B: Methodological* **64**, 24–40.

Kolak, M. and Anselin, L. (2020), 'A spatial perspective on the econometrics of program evaluation', *International Regional Science Review* **43(1-2)**, 128–153.

Vega, S. and Elhorst, J. P. (2015), 'The slx model', *Journal of Regional Science* **55(3)**, 339–363.