Forecasting The Yield Curves Using Macroeconomics Expectations and Time-Varying Volatility

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Abstract

This paper proposes a model of the term structure of interest rates that uses macroeconomic to model yield dynamics, and allow for time-varying volatility. Results suggest that the introduction of survey data on market participants' expectations improves significantly the out-of-sample forecasting performance of the model in terms of statistical measures of predictive accuracy. Additionally, we investigate the economic value of yield curve predictability based on an portfolio allocation exercise. Results indicate that modelling time-varying yield volatility is highly relevant and improves the economic relevance of forecasts regardless of the degree of risk aversion considered.

Keywords: Yield Curve Forecasting, Macroeconomics Expectations, Time-varying volatility, Dynamic Factor Models, Out-of-sample Forecasts

JEL C53, E43, G17

1. Introduction

The last two decades have seen mounting evidence of a strong link between macroeconomic factors and the yield curve[1]. Ang & Piazzesi (2003) incorporated macroeconomic variables under no-arbitrage restrictions and showed that models with macro factors computed from principal components outperformed models with only unobservable factors. Despite not considering out-of-sample forecasting, Diebold *et al.* (2006, henceforth DRA) find evidence that real activity, inflation, and the monetary policy instrument can explain a significant part of yield curve variations.

Additionally, several papers explore surveys of market expectations and broad macroeconomic information to link them with interest rates.[2] Altavilla *et al.* (2017), for instance, show that survey

[1] See, for example, Fleming & Remolona (2001); Piazzesi (2001, 2005); Dai & Philippon (2005); Diebold *et al.* (2005); Gallmeyer *et al.* (2005); Hördahl *et al.* (2006); Ang *et al.* (2006); Duffee (2006); Dewachter & Lyrio (2006); Dewachter *et al.* (2006); Rudebusch & Wu (2007, 2008); Bikbov & Chernov (2010); Bekaert *et al.* (2010); Duffee (2011); Gürkaynak & Wright (2012); Joslin *et al.* (2014).

[2] Ludvigson & Ng (2009); Piazzesi *et al.* (2009); Stark *et al.* (2010); Kim & Orphanides (2012); Chernov & Mueller

expectations can be exploited to improve the accuracy of yield curve forecasts. Moench (2008) jointly modeled the dynamics of macroeconomic variables and government bond yields in a dynamic factor model. He finds that information embedded in the macro factors helps provide out-of-sample yield forecasts that outperform the benchmark at intermediate and long horizons and for short and medium-term maturities. De Pooter *et al.* (2010) compare the forecast performance of several individual term structure models, they suggest that adding macroeconomic information improves interest rate forecasts, especially in and around recession periods. Fernandes & Vieira (2019) reveals that employing financial and macro information to build factors based on high-frequency forward-looking series in a factor-augmented DNS model can improve the predictive performance.

The gains obtained by including macroeconomic information depend on how macro information is incorporated in the model, as argued by Exterkate *et al.* (2013). Also, they suggest it is useful only for forecasting yields of medium-term maturities (between 1 and 5 years) and that factor-augmented methods perform well in relatively volatile periods, including the crisis period in 2008-2009 when simpler models do not suffice. Koopman & van der Wel (2013) also extending the class of dynamic factor yield curve models perform an out-of-sample forecasting study, and their results suggest that macroeconomic variables can lead to more accurate yield curve forecasts.[3]

The evidence exploring the links between macroeconomic variables and yield curve modeling for Brazil is not a novelty, notwithstanding the scarcity. Almeida & Faria (2014), for instance, evaluates the term structure forecasting as per Moench (2008) using common factors from macroeconomic series from January 2000 to May 2012. Their results suggest better predictive performance compared to the usual benchmarks but presented deterioration of the results with increased maturity. Also, by eliminating the no-arbitrage restrictions, they produced superior forecasting results. Vieira *et al.* (2017) show that the inclusion of forward-looking data set principal components improves the predictive ability of the factor-augmented VAR methodology with the Nelson-Siegel in out-of-sample analysis to the Brazilian term structure of interest rates. de Andrade Alves *et al.* (2023) investigates whether Brazilian Central Bank communication helps to forecast the yield curve. They include sentiment variables as additional factors into the dynamic Nelson-Siegel term structure model and found that these sentiment variables contain predictive information for yield curve forecasting.

Although the research cited above illustrated different approaches and evidences, they have some common features and hypotheses. One of them is constant volatility for all maturities throughout the sampling period. The second one, which may justify the first, is that the absence of time-varying volatility would not be a problem in estimation for advanced economies compared with emerging economies. The last one is a lack of evaluating forecasting results applied to an investment strategy. So, our goal is to explore these gaps to investigate whether modeling time-varying volatility in macro-term structure

(2012); Orphanides & Wei (2012); Ehling *et al.* (2018); Chun (2011); Favero *et al.* (2012); Chun (2012); Van Dijk *et al.* (2014); Altavilla *et al.* (2014).

[3] Poncela (2013) comments that what helps to increase the forecasting accuracy of the US term structure of interest rates is to restrict the transition matrix ϕ of the VAR(1) model proposed for the common factors. The forecasting results for the smooth dynamic factor model (SDFM) with and without macro variables are very similar, meaning that simpler models for the common factors (in the form of uncoupled factors) are preferred for forecasting.

models to an emerging market improves forecasting performance and whether these forecasts are worth it for an investor who cares about mean and variance.

Our analysis builds on contributions from Diebold *et al.* (2006) who use macroeconomic variables, and Koopman *et al.* (2010) that use a factor volatility structure for the latent variables with a specification based on GARCH models[4], both based on the Nelson & Siegel (1987) model of the term structure. In an application to Brazilian data, this paper provides evidence that adding backward- and forward-looking macroeconomic informations to the dynamic factor models with time-varying volatility can produce more accurate forecasts for forecasts of government bond yields. Following the recent literature that emphasizes the interaction between the yield curve and other economic variables, we present forecast results obtained from different model specifications, considering both with and without macroeconomic variables and time varying volatility.

Although our analysis is focused on statistical measures of predictive accuracy, it is important to evaluate the extent to which the apparent gains in predictive accuracy can be used in real time to improve investors' economic utility, that is, translate into better investment performance. Given that statistical significance does not necessarily imply economic significance, we follow what was done in Thornton & Valente (2012), Sarno *et al.* (2016), Caldeira *et al.* (2016), and Gargano *et al.* (2019), among others, and assess the economic value of the predictive power of interest rates by investigating the utility gains accrued to investors who exploit the predictability of yield curve relative to the benchmark model.

Our results confirm and extend results found in previous literature that add macroeconomic information.

2. Dynamic Factor Models for the Yield Curve

Dynamic factor models play a major role in econometrics since allowing the explanation of a large set of time series in terms of a small number of unobserved common factors (e.g Jungbacker *et al.*, 2014, and the literature therein). Many specifications for the yield curve can be viewed as dynamic factor models with a set of restrictions imposed on factor loadings (see, for example, Joslin *et al.*, 2013). In this section, we discuss the fourteen individual yield curve models beginning with the three-factor DNS model.

2.1. Dynamic Nelson-Siegel model

Diebold & Li (2006) estimate the term structure of interest rates using the model proposed by Nelson & Siegel (1987), assuming that the parameters vary over time. The following equation would describe the dynamics of the term structure

$$y_{i,t}(\tau_i) = \beta_{1,t} + \beta_{2,t}\left(\frac{1-e^{-\lambda\tau_i}}{\lambda\tau_i}\right) + \beta_{3,t}\left(\frac{1-e^{-\lambda\tau_i}}{\lambda\tau_i} - e^{-\lambda\tau_i}\right), \tag{1}$$

[4]Some ways to overcome this issue were proposed by (Bianchi *et al.*, 2009; Laurini & Hotta, 2010; Caldeira *et al.*, 2010; Hautsch & Yang, 2012)

where y_{it} denotes the yield at time t of a security with maturity τ_i, for $t = 1, \ldots, T$ and $i = 1, \ldots, N$, and λ is a decay parameter that can capture a variety of shapes of the yield curve through time, such as upward and downward sloping, and inversely humped. The $\beta_{1t}, \beta_{2t},$ and β_{3t} are time-varying parameters, or the state variables, that can be interpreted as the level, slope, and curvature latent components of the yield curve.

The DNS model is our starting point to model and forecast the yield curve. The dynamic movements or evolution of the yield curve factors, $\beta_{1t}, \beta_{2t},$ and β_{3t}, are assumed to follow a vector autoregressive process of first order, which allows for casting the yield curve latent factor model in state-space.

2.2. DNS in State-Space Representation

Diebold *et al.* (2006) note that DNS framework can be represented as a state space model by treating $\boldsymbol{\beta_t} = \beta_{j,t}$, for $j = 1, \ldots, 3$, as a latent vector. For these purpose, the general specification of the dynamic factor model is given by:

$$y_t = \Lambda(\lambda_t)\boldsymbol{\beta_t} + \boldsymbol{\varepsilon_t}, \qquad \boldsymbol{\varepsilon_t} \sim \mathcal{N}\left(0, \Sigma_\varepsilon\right), \tag{2}$$

where Λ is a $N \times K$ matrix of factor loadings, $\boldsymbol{\beta_t}$ is a K-dimensional stochastic process, and $\boldsymbol{\varepsilon_t}$ is the $N \times 1$ vector of measurement errors, whose covariance matrix given by Σ_ε. For any given, strictly positive λ_1, the $N \times K$ factor loading matrix $\Lambda(\lambda_t)$ is given by:

$$\Lambda_{ij}(\lambda_k) = \begin{cases} 1, & j = 1 \\ \psi_{i2} = \dfrac{1 - z_{1i}}{\lambda_1 \tau_i}, & j = 2 \\ \psi_{i3} = \dfrac{1 - z_{1i}}{\lambda_1 \tau_i} - z_{1i}, & j = 3, \end{cases}$$

where $z_{1,i} = \exp(-\lambda_1 \tau_i)$.

The state-space framework is achieved by assuming that the dynamic movements or evolution of the yield curve factors $\boldsymbol{\beta_t}$ are modeled by the following first-order vector-autoregressive process:

$$\boldsymbol{\beta_{t+1}} = \boldsymbol{\mu} + \Phi\left(\boldsymbol{\beta_t} - \boldsymbol{\mu}\right) + \eta_t, \qquad \eta_t \sim \mathcal{N}\left(0, \Sigma_\eta\right), \tag{3}$$

where μ is a $K \times 1$ vector of constants, Φ is a $K \times K$ coefficient matrix, and Σ_η is the covariance matrix of the disturbance vector η_t, which is independent of the vector of residuals $\boldsymbol{\varepsilon_t}$ for all t.

The matrix of variance-covariance of the innovations to the measurement system Σ_ε is assumed to be diagonal. This assumption implies that deviations of the observed yields of various maturities from those implied by the fitted yield curve are uncorrelated. While the matrix of variance-covariance of the innovations to the transition system Σ_η is unrestricted so that shocks to the three yield-curve factors are correlated.

2.3. The Factor-Augmented Nelson-Siegel Model

The first extension of the DNS model is the inclusion of macro-finance indicators in the model. In addition to the yield data, we have p factors available representing macroeconomic information available at the monthly frequency, covering the same period as the yield data. We include these factors in the state vector such that it becomes $(\beta_{1,t}, \beta_{2,t}, \beta_{3,t}, f_{1,t}, \ldots, f_{p,t})' = \boldsymbol{\beta}_t^{FA}$. With the extension of the state factor, the size of the coefficient matrix Φ^{FA} in the state equation increases from 3×3 to $(3+p) \times (3+p)$. The resulting state-space form is then given by

$$\mathbf{y}_t = \underbrace{\begin{bmatrix} \boldsymbol{\Lambda}(\lambda) & \mathbf{0}_{N \times p} \end{bmatrix}}_{\boldsymbol{\Lambda}(\lambda)^{FA}} \underbrace{\begin{bmatrix} \boldsymbol{\beta}_t \\ f_t \end{bmatrix}}_{\boldsymbol{\beta}_t^{FA}} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \boldsymbol{\Sigma}_\varepsilon), \tag{4}$$

$$\underbrace{\begin{bmatrix} \boldsymbol{\beta}_{t+1} \\ f_{t+1} \end{bmatrix}}_{\boldsymbol{\beta}_{t+1}^{FA}} = (I_{3+p} - \Phi^{FA}) \begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{0}_{p \times 1} \end{bmatrix} + \Phi^{FA} \underbrace{\begin{bmatrix} \boldsymbol{\beta}_t \\ f_t \end{bmatrix}}_{\boldsymbol{\beta}_t^{FA}} + \boldsymbol{\eta}_t^{FA}, \quad \boldsymbol{\eta}_t^{FA} \sim \mathcal{N}(\mathbf{0}_{3+p}, \boldsymbol{\Sigma}_\eta^{FA}) \tag{5}$$

for t = 1, . . . , T , where the dimensions of $\boldsymbol{\Phi}$, $\boldsymbol{\eta}_{t+1}$, and $\boldsymbol{\Sigma}_\eta$ are increased as appropriate. The coefficient matrix structure implies that the macro-factors affect the individual yields through the Nelson-Siegel factors and feedback from the yields to the macro-factors. Therefore, we estimated the DNS-Macro in that framework. In the following section, we explain the algorithm used in the estimation procedure.

2.4. Estimation Procedure

The estimation of the loading parameters λ in the measurement matrix in Eq. (2) is the key to estimating the state-space model. Keeping λ's fixed over the whole sample period, the equations (2) and (3) characterize a linear and Gaussian state-space model; thus, the Kalman filter can be used to obtain the likelihood function via the prediction error decomposition. The estimation procedures are as discussed below.

2.4.1. Estimation of Linear State Space Models Based on the Kalman Filter

Assuming that the decay parameters are constant, the measurement equation becomes linear. In this case, the DNS model is treated as linear Gaussian state-space models. Given the state-space formulation of the dynamic factor model presented in (2) and (3), the Kalman filter can be used to obtain the likelihood function via the prediction error decomposition. An optimization algorithm is used to maximize the likelihood function estimated by the Kalman filter, an iterative process of estimating and updating the measurement and transition equations until an optimal point is obtained. In short, the filter computes the optimal yields forecasts and the corresponding forecasting errors, after which the Gaussian likelihood function is evaluated using the prediction-error decomposition of the likelihood function for the forecasts and the states. It sequentially updates the measurement and transition equations until an optimal yield forecast is achieved.

Consider the general state-space representation in (2) and (3). This state-space model is estimated by applying a Kalman filter, a recursive formula running forwards through time to estimate latent factors from past observations. The Kalman filter evaluates the conditional means and variances of the latent factors $\boldsymbol{\beta}_{t+1}$ conditional on the information available up to and including time t, denoted as $\hat{b}_{t+1|t}$ and $P_{t+1|t}$ respectively. Using the transition equation in (3), the optimal predicted estimates is then given by

$$\hat{b}_{t+1|t} = \boldsymbol{\mu} + \Phi \left(b_{t|t} - \boldsymbol{\mu} \right), \tag{6}$$

$$P_{t+1|t} = \Phi P_t \Phi' + \Sigma_\eta, \tag{7}$$

where $P_{t+1|t}$ is mean square error (MSE), or covariance, matrix. Hence, the optimal filtered estimates $\hat{b}_{t+1}$ and P_{t+1} is given by

$$\hat{b}_{t+1} = \hat{b}_{t+1|t} + P_{t+1|t} \Lambda' F_{t+1|t}^{-1} v_{t+1}, \tag{8}$$

$$P_{t+1} = P_{t+1|t} - P_{t+1|t} \Lambda' F_{t+1|t}^{-1} P_{t+1|t}, \tag{9}$$

where $v_{t+1} = y_{t+1} - \Lambda \hat{b}_{t+1|t}$ is the prediction error, $F_{t+1|t} = \Lambda P_{t+1|t} \Lambda' + \Sigma_\varepsilon$ is the measurement prediction variance, and $P_{t+1|t} \Lambda' F_{t+1|t}^{-1}$ is called the Kalman gain.

The Kalman filter iterative process is initialized by using the unconditional mean and variance of $\boldsymbol{\beta}_t$. For this purpose, we carry out the 2-step procedure as described in Diebold & Li (2006). Specifically, the unconditional mean and covariance matrix of the state vector is started as follows

$$b_{1|0} = \mathbb{E}\left[\boldsymbol{\beta}_t\right] = \boldsymbol{\mu} \quad \text{and} \quad P_{1|0} = \mathbb{E}\left[\boldsymbol{\beta}_t \boldsymbol{\beta}_t'\right] = \Sigma_\beta,$$

where the unconditional covariance matrix of the state vector is the solution of $\Sigma_\beta - \Phi \Sigma_\beta \Phi = \Sigma_\eta$, which we can solve using the properties of the vectorization operator *vec* (see Christensen & van der Wel, 2019, for details).

Let the vector θ collects all unknown coefficients in the in the VAR parameter matrix Φ, variance matrices Σ_ε and Σ_η, and Λ and $\boldsymbol{\mu}$. To estimate the parameters vector θ, the likelihood function is constructed from the update step by assuming that the forecasting errors v_t are Gaussian. The Gaussian log-likelihood function is computed as

$$\ell(\theta) = -\frac{NT}{2} \log 2\pi - \frac{1}{2} \sum_{t=1}^{T} \log |F_t| - \frac{1}{2} \sum_{t=1}^{T} v_t' F_t^{-1} v_t. \tag{10}$$

As a result, $\ell(\theta)$ can be evaluated by Kalman filter for a given value of θ. By maximizing this log-likelihood function with respect to the parameters (collective represented as a vector θ) using a quasi-Newton optimization method results in maximum likelihood estimates of the parameters. The algorithm BFGS is used to maximize the log-likelihood function specified in (10) to obtain the estimates of the parameters θ.

Under this framework, we estimated the linear models. In the next sections, we extend the DNS model by considering conditional heteroskedasticity in the yield processes and treating the loading parameters as a stochastically time-varying latent factor.

2.4.2. Time-Varying Volatility

In the DNS model, we assume volatility is constant over time, which may be a restrictive assumption since yield curves are related to trading in the financial markets, then changes in volatility may occur. In general, heteroscedasticity is a constant problem in economics, especially in finance. The Kalman filter can not handle this problem, that is, the filter works under the hypothesis that the variance matrices are constant, or at least known. Assuming the GARCH structure, the matrix Σ_ε is time-varying.

To allow for conditional heteroscedasticity in the yield processes, we modify the DNS model by following Koopman *et al.* (2010), who propose capturing yield curve volatility allowing for a common variance component jointly affecting all individual yields. The common variance component is modeled as a generalized autoregressive conditional heteroscedasticity (GARCH) process. Harvey *et al.* (1992) already provides an extensive framework for incorporating this GARCH(1, 1) model into unobserved component time series models and how to deal with corresponding implications for estimation procedures. This factor can be interpreted as the volatility of an underlying bond market portfolio according to Engle & Ng (1993). The error in the measurement equation (2) are decomposed as

$$\boldsymbol{\varepsilon}_t = \Gamma_\varepsilon \varepsilon_t^* + \boldsymbol{\varepsilon}_t^\dagger, \quad t = 1, \dots, T, \tag{11}$$

where Z_ε and $\boldsymbol{\varepsilon}_t^\dagger$ are $N \times 1$ vectors of loadings and noise component respectively, and ε_t^* is a scalar representing the common disturbance term. The error components are mutually independent of each other and are distributed as follows

$$\varepsilon_t^* \sim \mathrm{NID}(0, h_t), \quad \text{and} \quad \boldsymbol{\varepsilon}_t^\dagger \sim \mathrm{NID}(\mathbf{0}, \Sigma_\varepsilon^\dagger), \quad t = 1, \dots T, \tag{12}$$

where $\Sigma_\varepsilon^\dagger$ is a diagonal matrix and h_t is the variance specified as a GARCH process, according to Bollerslev (1986). In this case, we have

$$h_{t+1} = \gamma_0 + \gamma_1 \varepsilon_t^{*2} + \gamma_2 h_t, \quad t = 1, \dots, T, \tag{13}$$

and the estimated parameters have the constraints $\gamma_0 > 0$, $0 < \gamma_1 < 0$, $0 < \gamma_2 < 0$, $h_1 = \gamma_0(1 - \gamma_1 - \gamma_2)^{-1}$, and $(\gamma_1 + \gamma_2) < 1$. The weights vector Γ_ε can be normalized to avoid identification problems, such that $\Gamma_\varepsilon' \Gamma_\varepsilon = 1$, but we follow Koopman *et al.* (2010) and fixed γ_0 at 1×10^{-4}. The resulting time-varying variance matrix for $\boldsymbol{\varepsilon}_t$ is given by

$$\Sigma_\varepsilon(h_t) = h_t \Gamma_\varepsilon \Gamma_\varepsilon' + \Sigma_\varepsilon^\dagger, \tag{14}$$

where $\Sigma_\varepsilon(h_t)$ depends on a single factor described by the GARCH process in (13). The (unconditional) time-varying variance matrix of y_t is $\Lambda(\lambda)\Sigma_\beta\Lambda(\lambda)' + \Sigma_\varepsilon(h_t)$, where Σ_β is the solution of $\Sigma_\beta - \Phi\Sigma_\beta\Phi' = \Sigma_\eta$. The GARCH factor ε_t^* is incorporated in the measure equation (2), which is treated as a latent factor. Hence, we include ε_t^* in the state vector alongside the DNS factors.

The resulting observation and state equations of the DNS models with time-varying volatility can be rewritten into the state-space formulation as

$$y_t = \begin{bmatrix} \Lambda(\lambda) & \Gamma_\varepsilon \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_t \\ \varepsilon_t^* \end{bmatrix} + \varepsilon_t^\dagger, \quad \varepsilon_t^\dagger \sim \mathscr{N}(0, \Sigma_\varepsilon^\dagger), \tag{15}$$

$$\begin{bmatrix} \boldsymbol{\beta}_{t+1} \\ \varepsilon_{t+1}^* \end{bmatrix} = \begin{bmatrix} (I_j - \Phi_j)\boldsymbol{\mu} \\ 0 \end{bmatrix} \begin{bmatrix} \Phi_j & \mathbf{0}_{j\times 1} \\ \mathbf{0}_{1\times j} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_t \\ \varepsilon_t^* \end{bmatrix} + \begin{bmatrix} \boldsymbol{\eta}_t \\ \varepsilon_{t+1}^* \end{bmatrix}, \qquad \begin{bmatrix} \boldsymbol{\eta}_t \\ \varepsilon_{t+1}^* \end{bmatrix} \sim \mathscr{N}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \Sigma_\eta & \mathbf{0}_{j\times 1} \\ \mathbf{0}_{1\times j} & h_{t+1} \end{bmatrix} \right), \tag{16}$$

for $t = 1,\ldots, T$. The addition of GARCH disturbances and extra parameters requires applying certain adjustments to the estimation procedure.

Since h_{t+1} in (13) is a function of its past values and unobserved values of ε_t^*, it is not possible to calculate the values required for h_{t+1} at time t. Specifically, Harvey *et al.* (1992) explains that, although the models are not conditionally Gaussian because knowledge of past observations does not imply knowledge of past GARCH errors, we may treat the models as though they are conditionally Gaussian. Because of that, in the presence of GARCH errors, the Kalman filter can be regarded as a quasi-optimal filter instead of optimal. Harvey *et al.* (1992) propose to take the expectation of the latent term in the volatility specification such that we obtain an estimate for h_{t+1}, given by

$$\hat{h}_{t+1|t} = \gamma_0 + \gamma_1 \mathbb{E}\left[\varepsilon_t^{*2} | \mathfrak{I}_t\right] + \gamma_2 \hat{h}_{t|t-1}, \quad t = 1, \ldots, T, \tag{17}$$

where $\mathfrak{I}_t$ denotes all information available up to and including time t. To calculate the expectation term we note that

$$\varepsilon_t^* = \mathbb{E}\left[\varepsilon_{t-1} | \mathfrak{I}_t\right] + \left(\varepsilon_t^* - \mathbb{E}\left[\varepsilon_t^* | \mathfrak{I}_t\right]\right).$$

By squaring and taking conditional expectations we can shown that

$$\begin{aligned} \mathbb{E}\left[{\varepsilon_t^*}^2 | \mathfrak{I}_t\right] &= \mathbb{E}\left[\varepsilon_t^* | \mathfrak{I}_t\right]^2 + \mathbb{E}\left[\left(\varepsilon_t^* - \mathbb{E}\left[\varepsilon_t^* | \mathfrak{I}_t\right]\right)^2\right], \\ &= {\hat{\varepsilon}_{t|t}^*}^2 + P_{t|t}^\varepsilon, \end{aligned} \tag{18}$$

where $\hat{\varepsilon}_{t|t}^*$ is the last element of the filtered state $b_{t|t}$ and $P_{t|t}^{\varepsilon^*}$ is the last diagonal element of the $P_{t|t}$, the filtered variance of $\hat{\varepsilon}_{t|t}^*$. Then, we substitute the expression $\mathbb{E}\left[{\varepsilon_t^*}^2 | \mathfrak{I}_t\right]$ into (17) to obtain a prediction for the volatility component h_{t+1}. Lastly, we insert the predicted value h_{t+1} in the position (j, j) of the variance matrix Σ_η, corresponding to the location of ε_t^* in the state vector. With this framework we estimated the DNS-GARCH and extensions.

3. Out-of-Sample Analysis

In order to forecasts h-months ahead, the steps below follow after the filtering step and estimation of the optimal set of parameters $\boldsymbol{\theta}$ throughout the sample, see Durbin & Koopman (2012) for more details.

Therefore, we have the following

$$\begin{aligned}\mathbf{y}_{t+1} &= \boldsymbol{\Lambda}(\lambda)\mathbb{E}(\boldsymbol{\beta}_{t+1}|Y_t),\\ \bar{\mathbf{y}}_{t+1} &= \boldsymbol{\Lambda}(\lambda)\bar{\mathbf{b}}_{t+1},\end{aligned} \tag{19}$$

where $\bar{\mathbf{b}}_{t+1}$ is the state vector and $\bar{\mathbf{B}}_{t+1}$ the variance matrix of the states calculated by the Kalman filter in (6) and (7). For other forecasts, the filter can be rewritten to $h = 2, \ldots, H$, as follows

$$\bar{\mathbf{b}}_{t+h} = \boldsymbol{\mu} + \boldsymbol{\Phi}(\bar{\mathbf{b}}_{t+1} - \boldsymbol{\mu}), \tag{20}$$

$$\bar{\mathbf{B}}_{t+h} = \boldsymbol{\Phi}\bar{\mathbf{B}}_{t+1}\hat{\boldsymbol{\Phi}}' + \boldsymbol{\Sigma}_{\eta}, \tag{21}$$

$$\bar{\mathbf{y}}_{t+h} = \boldsymbol{\Lambda}(\lambda)\bar{\mathbf{b}}_{t+h}, \tag{22}$$

where the states vector and the variance matrix of the previous estimation states are used to calculate the predictions in the step $h+1$. With the time-varying loading parameter, the difference for predictions lies in the loading matrix that multiplies the state vector, that is, $\mathbf{Z}_t(\mathbf{a}_{t|t-1})$ instead of $\boldsymbol{\Lambda}(\lambda)$ in (19) and (22).

The out-of-sample predictions are assessed by the relative sizes of the root mean square error (RMSPE) of all considered models relative to those from the DNS baseline model. The RMSPE is calculated as follows:

$$R(h, \tau) = \sqrt{\frac{1}{n}\sum_t [\hat{y}_{t+h|t}(\tau) - y_{t+h}(\tau)]^2}, \tag{23}$$

where n is the number of forecasts previously defined in 312. The drawback of using RMSPE is that this is a single statistic summarizing individual forecasting errors over an entire sample. Although often used, they do not give any insight as to where in the sample a particular model makes its largest and smallest forecast errors. Therefore, we also graphically analyze the cumulative squared forecast errors (CSFE) proposed by Welch & Goyal (2008). These cumulative prediction errors series clearly depicts when a model outperforms or underperforms a given benchmark and could motivate the use of adaptive forecast combination schemes. The CSFE is given by:

$$\mathrm{CSFE}_m(h, \tau) = \sum_t \left[\left(\hat{y}_{t+h|t,\mathrm{bench}}(\tau) - y_{t+h}(\tau)\right)^2 - \left(\hat{y}_{t+h|t,m}(\tau) - y_{t+h}(\tau)\right)^2\right]. \tag{24}$$

In the case a model outperforms the benchmark, the $\mathrm{CSFE}_m(h, \tau)$ will be an increasing series. If the benchmark produces more accurate forecasts, then $\mathrm{CSFE}_m(h, \tau)$ will tend to be decreasing.

We use the Giacomini & White (2006) test to assess whether the forecasts of two competing models are statistically different. The Giacomini-White (GW) test is a test of conditional forecasting ability and is constructed under the assumption that forecasts are generated using a moving data window. This is a test of equal forecasting accuracy and as such can handle forecasts based on both nested and non-nested models, regardless from the estimation procedures used in the derivation of the forecasts.

Lastly, we implement the Model Confidence Set (MCS), approach developed by Hansen *et al.* (2011), which consist on a sequence of tests which permits to construct a set of 'superior' models, where the null hypothesis of Equal Predictive Ability (EPA) is not rejected at a certain confidence level. The EPA statistic tests is calculated for an arbitrary loss function, in our case we test squared errors of DNS model against competing models.

3.1. The Economic Value of the Yield Curve Predictability

Although our analysis is focused on statistical measures of predictive accuracy, it is important to evaluate the extent to which the apparent gains in predictive accuracy can be used in real time to improve investors' economic utility, that is, translate into better investment performance. Given that statistical significance does not necessarily imply economic significance (Thornton & Valente, 2012; Sarno *et al.*, 2016; Caldeira *et al.*, 2016; Gargano *et al.*, 2019), we assess the economic value of the predictive power of interest rates by investigating the utility gains accrued to investors who exploit the predictability of yield curve relative to a no-predictability alternative associated with the random-walk model.

In this section, we explore the empirical evidence linking statistical forecasting evaluation with economic utility. To this purpose, we consider a mean-variance investor with quadratic utility and relative risk aversion γ who allocates her portfolio on a risky bond with τ periods to maturity versus a one-month T-bill that pays the risky free rate (Rapach & Zhou, 2013). At the end of t, the investor allocates the following share of her portfolio to bond with maturity τ_i during $t+1$:

$$w_{i,t} = \left(\frac{1}{\gamma}\right)\left(\frac{\hat{r}_{t+h}^{(\tau_i)}}{\hat{\sigma}_{t+h}^{2,(\tau_i)}}\right) \tag{25}$$

where $\hat{r}_{t+h}^{(\tau_i)} = \tau_i y_t^{\tau_i} - (\tau_i - h)\hat{y}_{t+h}^{\tau_i - h}$ is a return forecast for the bond with maturity τ_i in time t and $\hat{\sigma}_i^2$ is a forecast of the variance[5] of bond returns to models with constant volatility and $\varSigma_{r_{t|t-1}} = \tau'\tau \otimes [\varLambda(\lambda)\varSigma_\beta\varLambda(\lambda)' + \varSigma_\varepsilon(h_t)]$ to time-varying volatility. Over the forecast evaluation period, the investor realizes the average utility,

$$\hat{\nu}_i = \hat{\mu}_i - 0.5\gamma\hat{\sigma}_i^2, \tag{26}$$

where $\hat{\mu}_i\,(\hat{\sigma}_i^2)$ is the sample mean (variance) of the portfolio formed on the basis of $\hat{r}_{t+h}^{(\tau_i)}$ and $\hat{\sigma}_i^2$ over the forecast evaluation period. The resulting sequences of allocation weights are next used to calculate realized utilities. For each model m, the realized utility are converted into equivalent returns CER, i.e., the difference between utility (26) with model m and the DNS represents the utility gain accruing to using the competitors models forecast of the bond yields in place of the DNS benchmark forecast in the asset allocation decision. This utility gain (certainty equivalent return) can be interpreted as the portfolio management fee that an investor would be willing to pay to have access to the information in the model forecast relative to the information in the benchmark DNS model.

[5] We follow the strategy of Rapach & Zhou (2013) and estimate the variance of bond returns using the sample variance computed from a one-year (252-obs) rolling window of historical returns.

4. Data and Results

4.1. Data

This paper's data set consists of monthly closing prices observed for yields of future DI contracts. Based on the observed rates for the available maturities, the data were converted to fixed maturities of 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 36, 42, 48, and 60 months, through interpolations using cubic splines. The database contains the maturities with the highest liquidity for January 2003 through December 2019 (T = 204 observations) and represents the most liquid DI contracts negotiated during the analyzed period. We assess the model's performance by splitting the sample into two parts: the first one includes 132 observations used to estimate all models' parameters. The second part is used to analyze the performance out-of-sample of bond portfolios obtained from the model, with 72 observations.

Table 1 displays the descriptive statistics for the Brazilian interest rate curve. For each of the 14 time series, we report average, standard deviation, minimum, maximum, and the last three columns contain sample autocorrelations at displacements of 1, 6, and 12 months. Descriptive statistics presented in Table 1 seem to confirm key stylized facts about yield curves: the sample average curve is upward sloping and concave, volatility is decreasing with maturity, autocorrelations are very high and increasing with maturity. Also, there is a high persistence in the yields: the first-order autocorrelation for all maturities is above 0.87 for each maturity.

Figure **??** presents a three-dimensional plot of the data set and illustrates how yield levels and spreads vary substantially throughout the sample. Although the yield series change heavily over time for each of the maturities, a strong common pattern in the 14 series over time is apparent. The sample contains 204 monthly observations with maturities of τ = 3, 6, 9, 12, 15, 18, 21 24, 27, 30, 36, 39, 48, and 60 months.

We use three macroeconomic factors: the Selic interest rate is the monetary policy interest rate, i.e., the key tool used by the Central Bank of Brazil (BCB) to implement the monetary policy. The Selic rate, or 'over Selic', is the Brazilian federal funds rate. Precisely, Selic rate is the weighted average interest rate of the overnight interbank operations – collateralized by federal government securities – carried out at the Special System for Settlement and Custody (Selic).

Under the inflation-targeting regime, the BCB's Monetary Policy Committee (Copom) regularly sets the target for the Selic rate. Within the relevant horizon for the monetary policy, the Copom aims to keep the Extended National Consumer Price Index (IPCA inflation rate) around the target and anchor inflation expectations. Accordingly, the BCB performs daily open market operations to keep the effective Selic rate at the target set by Copom.

Brazil has several price indexes that differ significantly in scope, depending on their particular purposes. A price index can be designed to reflect the cost of living for a specific group of households, but each household will have its price index based on its consumer basket. In this sense, there can be different inflation perceptions between what the citizen notices as inflation and the variation of several price indexes. The inflation index adopted here is the IPCA which is the reference for the Brazilian inflation-targeting system. The BCB ensures that the IPCA's annual inflation is centered at the inflation target set by the National Monetary Council (CMN).

Table 1: **Descriptive statistics for the term structure of interest rates**

The table reports summary statistics for Brazil yield curve over the period 2003-2019. We examine monthly data, constructed using the spline method. For each maturity we show mean, standard deviation, skewness, raw kurtosis, minimum, maximum, and three auto-correlations coefficients, $\hat{\rho}_1$, $\hat{\rho}_6$, $\hat{\rho}_{12}$. Also the table reports proxy estimates for level, slope, and curvature of the yield curve. The proxies are defined as follows: for level, the highest maturity bond (60 months); for slope, the difference between the bond of 60 months and the bond of 3 months; and for curvature, two times the bond of 18 months minus the sum of bond of 3 months and bond of 60 months.

Maturity	Mean	Std Dev	Min	Max	Skewness	Kurtosis	Acf		
							$\hat{\rho}_1$	$\hat{\rho}_6$	$\hat{\rho}_{12}$
M3	12.19	4.52	4.30	27.50	0.90	4.11	0.96	0.70	0.48
M6	12.20	4.47	4.30	28.30	0.90	4.24	0.96	0.69	0.48
M9	12.24	4.42	4.40	29.00	0.92	4.56	0.95	0.68	0.48
M12	12.30	4.39	4.50	29.60	0.98	4.93	0.94	0.67	0.47
M15	12.38	4.34	4.50	30.40	1.06	5.39	0.94	0.65	0.46
M18	12.48	4.31	4.60	31.30	1.16	5.95	0.93	0.64	0.45
M21	12.57	4.28	4.70	32.30	1.29	6.64	0.93	0.62	0.44
M24	12.65	4.27	4.90	33.40	1.43	7.41	0.92	0.61	0.43
M27	12.74	4.25	5.00	34.30	1.57	8.15	0.92	0.60	0.42
M30	12.80	4.26	5.10	35.10	1.69	8.86	0.91	0.59	0.41
M36	12.92	4.28	5.30	36.70	1.95	10.35	0.90	0.57	0.40
M42	13.02	4.34	5.60	38.40	2.20	11.93	0.90	0.55	0.38
M48	13.10	4.38	5.70	39.40	2.37	12.96	0.89	0.54	0.38
M60 (**Level**)	13.18	4.35	6.00	39.40	2.44	13.12	0.89	0.55	0.38
Slope	0.99	2.10	−3.90	11.90	0.75	6.77	0.84	0.35	0.01
Curvature	−0.24	1.17	−3.00	3.10	0.18	2.86	0.87	0.40	0.08

IBC-Br is an indicator of the monthly periodicity, which incorporates the pathway of the variables considered proxies to economic sectors such as Agriculture and livestock, Industry, and Services. The well-known adherence of the trajectory of the IBC-Br to the GDP behavior confirms the importance of monitoring the indicator to understand better and anticipate the activity analysis.

We use market expectations factors from the BCB's Market Expectations System, which monitors market expectations regarding the main macroeconomic variables, providing important inputs for the monetary policy decision-making process.

The BCB carries out the "Focus Survey", compiling forecasts of about 140 banks, asset managers, and other institutions (real sector companies, brokers, consultancies, etc.). The Survey daily monitors the market expectations for several inflation indices, the GDP and industrial production growth, the exchange rate, the Selic rate, fiscal indicators, and external sector variables. Based on this Survey, the BCB compiles daily – and releases weekly – the Focus Market Readout, which brings the summary of the statistics calculated over the information collected. We use the IPCA inflation accumulated median percent change, Over-Selic Target median percent p.y., and total GDP median percent change over the next 12 months (see Figure 1).

5. Results

We use a rolling estimation window of 72 monthly observations (i.e., six years) for computing our results. We produce forecasts for 1-month, 3-month, 6-month, and 12-month-ahead. To compare the performance of out-of-sample forecasts, we compute the root mean square forecast error (RMSFE). Moreover, the Giacomini & White (2006) test (GW-test) is used to assess whether each model outperforms the DNS. Table 2 report statistical measures of the out-of-sample forecasting performance at various horizons. The first row of entries in each panel of the tables report the value of RMSFE (expressed in basis points) for the DNS model, while all other rows report statistic relative to the DNS.

Table 2 reports out-of-sample forecast performance measured in terms of RMSPE. This table is divided in four panels A through D, each corresponding to a different forecast horizon $(1, 3, 6$ and 12 steps ahead).The first row in each panel contains the RMSPE of the DNS baseline forecasts, whereas the remaining rows report RMSPE of a given model relative to those of the benchmark. Therefore any number below one indicates outperformance relative to the benchmark, whereas any number larger than one indicates underperformance. Asterisks to the right of entries indicate that, at the 10% level of significance, the null hypothesis of the GW test is not rejected. Bold type indicates that the model belongs to $\widehat{M}^*_{0.75}$, the set of superior models containing the best models with probability no less than 75%.

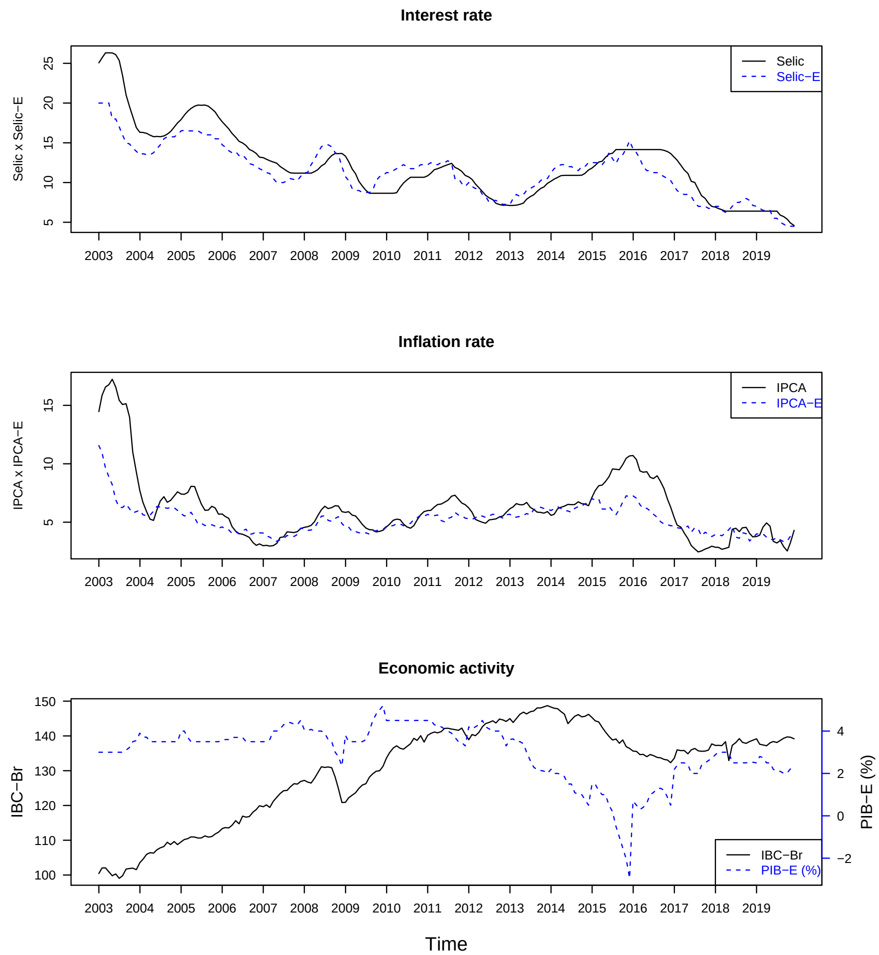


Figure 1: This figure shows the SELIC interest rate, the Extended National Consumer Price Index (IPCA), and Index of Economic Activity of the Central Bank (IBC-Br) in solid lines and his expectations (GDP for IBC-Br) of one year in dashed lines, respectively. The sample contains 194 monthly observations from January 2003 through December 2019 for realized series and from 2002 through 2018 for expectations.

Table 2: **Relative Root Mean Squared Forecast Errors**

The Table reports the relative root mean squared forecast errors (RMSFE) relative to the Random-Walk model (RW) model for the 1-month, 3-months, 6-months, and 12-months forecast horizons. The evaluation sample is 2014:1 to 2019:12 (73 out-of-sample forecasts). The first line in each panel of the table reports the value of RMSFE (expressed in basis points) for the RW model, while all other lines reports statistics relative to the RW. The following model abbreviations are used in the table: DNS-Macro model for realized macroeconomic factors , DNS-MacroE1 model for market expectations of macroeconomic factors, and the last three models have time-varying volatility (-GARCH). Numbers smaller than one indicate that models outperform the RW, whereas numbers larger than one indicate underperformance. The * on the right of the cell entries indicate the level at which the Giacomini & White (2006) test rejects the null of equal forecasting accuracy at least 10% level. Shaded values indicate that the model belong to Model Confidence Set (MCS) Hansen *et al.* (2011).

Model	**Maturity**													
	M3	M6	M9	M12	M15	M18	M21	M24	M27	M30	M36	M42	M48	M60
Panel A: 1-month ahead forecasts														
RW	41.26	39.69	42.97	47.79	51.04	54.47	56.47	57.74	58.80	60.69	61.63	63.07	64.53	66.56
DNS	0.933	0.992	0.999	0.995	0.982	0.972	0.971	0.963	0.960	0.959	0.967	0.966	0.972	0.971
DNS – Macro	0.940	1.010	0.989	0.976	0.970	0.971	0.977	0.976	0.978	0.978	0.992	0.991	0.996	0.995
DNS – MacroE1	0.881	0.892*	0.978	1.009	1.010	1.009	1.018	1.016	1.020	1.019	1.033	1.034	1.049	1.063
DNS – GARCH	0.964	0.978	1.040	1.056	1.055	1.044	1.050	1.048	1.048	1.042	1.045	1.044	1.047	1.045
DNS – GARCH – Macro	0.940	1.010	0.989	0.976	0.970	0.971	0.976	0.976	0.978	0.978	0.992	0.991	0.996	0.995
DNS – GARCH – MacroE1	0.881	0.892*	0.979	1.009	1.010	1.009	1.018	1.016	1.020	1.019	1.034	1.034	1.049	1.063
Panel B: 3-months ahead forecasts														
RW	90.21	90.41	94.52	99.69	103.88	108.63	111.44	112.61	114.64	117.00	120.34	121.97	124.01	125.32
DNS	0.824	0.892	0.934	0.959	0.953	0.941	0.932	0.925	0.914	0.905	0.887*	0.880*	0.878*	0.878*
DNS – Macro	0.795	0.877	0.913	0.938	0.937	0.935	0.936	0.936	0.932	0.928	0.920*	0.920*	0.922*	0.929*
DNS – MacroE1	0.668	0.800	0.877	0.93	0.942	0.945	0.953	0.955	0.951	0.947	0.934	0.935	0.939	0.952
DNS – GARCH	0.818	0.929	0.977	0.997	0.999	0.993	0.992	0.996	0.992	0.985	0.972	0.974	0.974	0.981
DNS – GARCH – Macro	0.795	0.877	0.913	0.938	0.937	0.935	0.936	0.936	0.932	0.928	0.920*	0.920*	0.922*	0.929*
DNS – GARCH – MacroE1	0.668	0.800*	0.877	0.930	0.942	0.945	0.953	0.956	0.951	0.947	0.934	0.935	0.939	0.952
Panel C: 6-months ahead forecasts														
RW	168.06	166.53	167, 34	169.92	172.07	174.62	176.65	177.09	178.23	180.37	184.08	186.97	187.46	188.79
DNS	0.760	0.835	0.892	0.922	0.926	0.919	0.907	0.896	0.884	0.868	0.834	0.815	0.806	0.790
DNS – Macro	0.722	0.819	0.886	0.925	0.936	0.939	0.935	0.930	0.924	0.914	0.889	0.877	0.873	0.864
DNS – MacroE1	0.669	0.782	0.858	0.906	0.922	0.928	0.929	0.926	0.923	0.914	0.885	0.873	0.867	0.86
DNS – GARCH	0.849	0.916	0.949	0.966	0.97	0.965	0.958	0.956	0.948	0.936	0.911	0.903	0.899	0.895
DNS – GARCH – Macro	0.722	0.819	0.886	0.925	0.936	0.939	0.935	0.930	0.924	0.914	0.889	0.877	0.873	0.864
DNS – GARCH – MacroE1	0.669	0.782	0.858	0.906	0.923	0.928	0.929	0.926	0.923	0.914	0.885	0.873	0.867	0.860
Panel D: 12-months ahead forecasts														
RW	296.67	294.33	290.84	286.67	281.94	277.09	272.77	269.43	267.11	265.25	264.99	263.29	262.12	260.45
DNS	0.823	0.875	0.914	0.940	0.951	0.954	0.951	0.941	0.929	0.918	0.884	0.861	0.844	0.819
DNS – Macro	0.830	0.924	0.991	1.036	1.059	1.072	1.073	1.068	1.057	1, 0479	1.014	0.990	0.973	0.944
DNS – MacroE1	0.869	0.944	0.996	1.031	1.044	1.047	1.045	1.034	1.019	1.006	0.966	0.937	0.917	0.885
DNS – GARCH	0.894	0.923	0.951	0.969	0.976	0.982	0.977	0.970	0.959	0.951	0.924	0.909	0.899	0.886
DNS – GARCH – Macro	0.830	0.924	0.991	1.036	1.059	1.072	1.074	1.068	1.057	1.048	1.014	0.990	0.973	0.944
DNS – GARCH – MacroE1	0.869	0.944	0.996	1.031	1.044	1.048	1.045	1.034	1.019	1.006	0.966	0.938	0.917	0.885

To analyze the accuracy of the forecasts in different time intervals, we follow **?** and plot the difference in cumulative square forecast errors between each of the prediction models and the RW along the out-of-sample evaluation period.

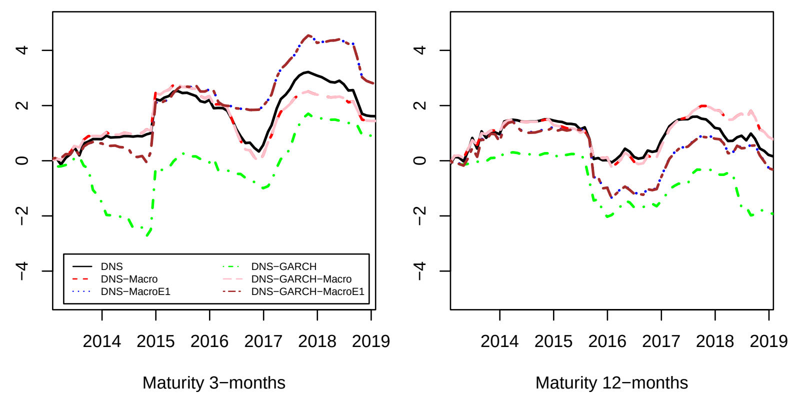
5.1. Economic Evaluation Results

Table 3 reports certainty equivalent (average utility gains in annualized percent return) for a mean-variance investor with $\gamma = \{0.1, 0.5, 1, 5\}$ who allocates among 1 to 5 years bonds and risk-free rate using forecasts based on competitors models in place of DNS forecasts.

Figure 2: **Cumulative squared forecast errors (1- and 3-months ahead)**

Note: Figures show the cumulative squared forecast errors (CSFE) of Nelson-Siegel Extensions relative to the DNS baseline model. Figure shows CSFEs for a 1- and 6-month forecast horizon. The evaluation sample is from January 2014 through December 2019 (73 out-of-sample forecasts).

(a) One-month ahead



(b) Six-month ahead

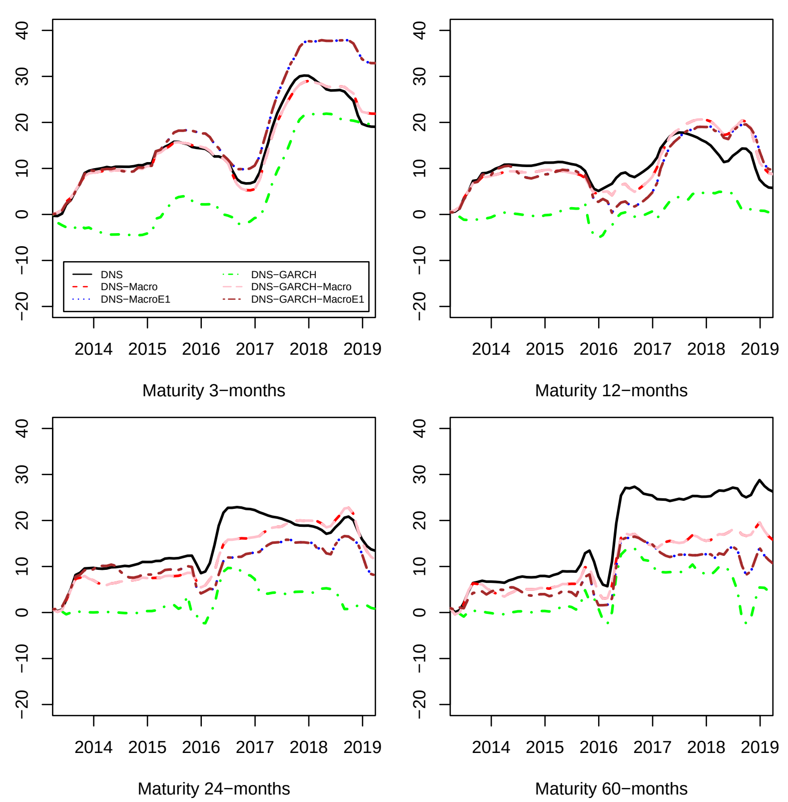
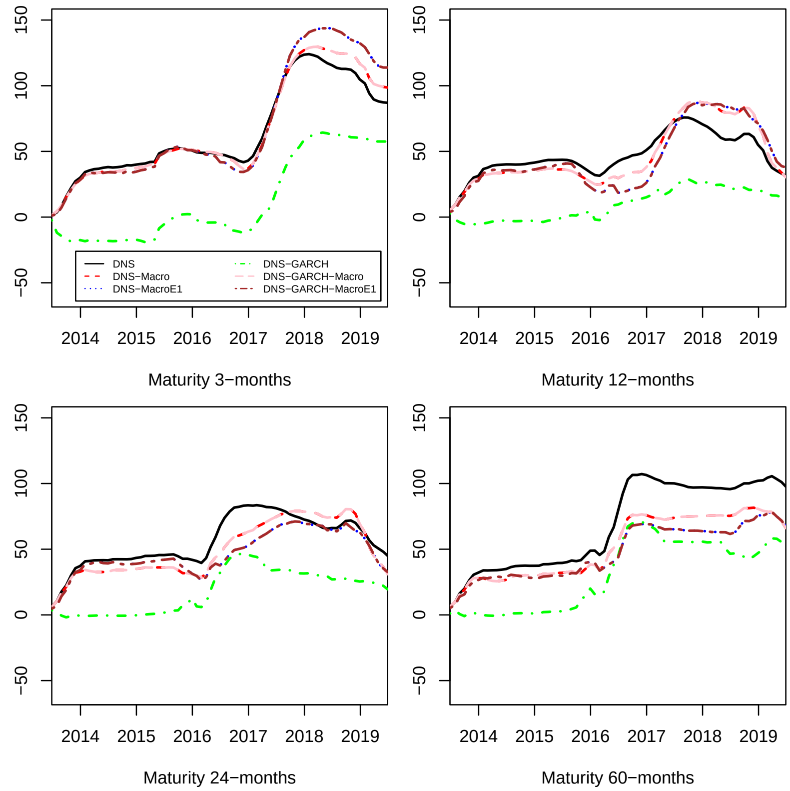


Figure 3: **Cumulative squared forecast errors (6- and 12-months ahead)**

Note: Figures show the cumulative squared forecast errors (CSFE) of Nelson-Siegel Extensions relative to the DNS baseline model. Figure shows CSFEs for a 1- and 6-month forecast horizon. The evaluation sample is from January 2014 through December 2019 (73 out-of-sample forecasts).

(a) Twelve-month ahead



(b) Six-month ahead

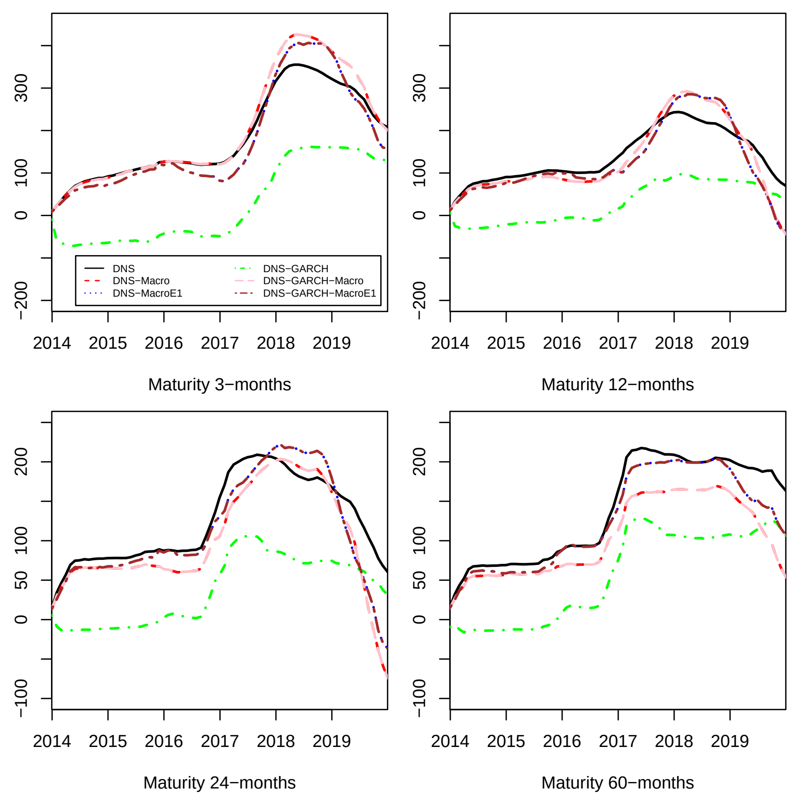


Table 3: **Out-of-sample economic evaluation of the yield curve forecasting from 2011:01 to 2017:04**

Note: This table reports average utility gain (δ) the portfolio management fee (in annualized percent return) that an investor with mean-variance preferences and risk aversion coefficient of 0.1 to 5 would be willing to pay to have access to the forecasting method relative to the DNS benchmark forecast. The following model abbreviations are used in the table: DNS-Macro model for realized macroeconomic factors, DNS-MacroE1 model for market expectations of macroeconomic factors, and the last three models have time-varying volatility (-GARCH). The sample starts on January 2000 and the evaluation period is January 2011 to June 2017.

Model	$\gamma = 0.1$					$\gamma = 0.5$					$\gamma = 1$					$\gamma = 5$				
	$\tau = \mathbf{1}$	$\tau = \mathbf{2}$	$\tau = \mathbf{3}$	$\tau = \mathbf{4}$	$\tau = \mathbf{5}$	$\tau = \mathbf{1}$	$\tau = \mathbf{2}$	$\tau = \mathbf{3}$	$\tau = \mathbf{4}$	$\tau = \mathbf{5}$	$\tau = \mathbf{1}$	$\tau = \mathbf{2}$	$\tau = \mathbf{3}$	$\tau = \mathbf{4}$	$\tau = \mathbf{5}$	$\tau = \mathbf{1}$	$\tau = \mathbf{2}$	$\tau = \mathbf{3}$	$\tau = \mathbf{4}$	$\tau = \mathbf{5}$
horizon = 1-month ahead																				
DNS	197.70	2.403	−0.279	−0.901	−0.643	50.16	1.843	0.516	0.138	0.072	31.72	1.773	0.615	0.268	0.161	16.97	1.717	0.695	0.372	0.233
DNS − Macro	226.09	7.118	1.431	−0.296	−0.504	54.65	2.587	0.786	0.234	0.094	33.22	2.021	0.705	0.300	0.169	16.07	1.568	0.641	0.353	0.229
DNS − MacroE1	213.77	8.047	3.377	1.657	1.286	52.70	2.734	1.093	0.542	0.377	32.57	2.07	0.808	0.403	0.263	16.46	1.538	0.579	0.291	0.172
DNS − GARCH	449.44	34.41	12.92	6.247	3.711	89.91	6.896	2.600	1.267	0.760	44.97	3.457	1.31	0.644	0.391	9.019	0.706	0.278	0.146	0.095
DNS − GARCH − Macro	449.54	34.48	12.98	6.272	3.701	89.93	6.907	2.609	1.271	0.758	44.98	3.461	1.313	0.646	0.39	9.016	0.704	0.276	0.145	0.096
DNS − GARCH − MacroE1	449.41	34.40	12.92	6.258	3.726	89.91	6.895	2.601	1.268	0.762	44.97	3.457	1.310	0.645	0.391	9.020	0.706	0.278	0.146	0.095
horizon = 3-month ahead																				
DNS	38.21	7.202	4.000	2.762	2.271	13.71	2.135	1.054	0.673	0.512	10.65	1.501	0.685	0.412	0.292	8.204	0.995	0.391	0.203	0.116
DNS − Macro	54.28	10.606	5.521	3.529	2.668	16.25	2.672	1.294	0.794	0.574	11.50	1.680	0.765	0.452	0.313	7.697	0.887	0.343	0.179	0.103
DNS − MacroE1	47.23	10.08	5.635	3.899	3.122	15.14	2.590	1.312	0.852	0.646	11.13	1.653	0.771	0.472	0.337	7.919	0.904	0.339	0.167	0.089
DNS − GARCH	181.10	23.13	9.533	5.22	3.377	36.28	4.649	1.927	1.061	0.686	18.17	2.34	0.977	0.541	0.350	3.692	0.492	0.216	0.125	0.081
DNS − GARCH − Macro	181.30	23.25	9.65	5.309	3.423	36.31	4.668	1.946	1.075	0.694	18.18	2.346	0.983	0.546	0.353	3.685	0.488	0.212	0.122	0.080
DNS − GARCH − MacroE1	181.04	23.17	9.609	5.326	3.497	36.27	4.656	1.939	1.078	0.705	18.17	2.342	0.981	0.547	0.356	3.694	0.490	0.214	0.122	0.077
horizon = 6-month ahead																				
DNS	12.64	14.251	8.698	6.667	5.362	5.676	2.809	1.637	1.218	0.953	4.806	1.379	0.754	0.537	0.402	4.110	0.235	0.048	−0.008	−0.039
DNS − Macro	20.07	15.762	9.355	6.940	5.422	6.850	3.048	1.741	1.261	0.963	5.197	1.459	0.789	0.551	0.405	3.875	0.187	0.028	−0.017	−0.041
DNS − MacroE1	17.61	15.845	9.206	6.741	5.257	6.461	3.061	1.717	1.23	0.937	5.067	1.463	0.781	0.541	0.397	3.952	0.185	0.032	−0.010	−0.036
DNS − GARCH	85.76	13.442	6.843	4.953	4.096	17.22	2.682	1.344	0.947	0.753	8.654	1.337	0.657	0.447	0.335	1.800	0.261	0.107	0.046	0.001
DNS − GARCH − Macro	85.96	13.50	6.872	4.954	4.031	17.25	2.691	1.349	0.947	0.743	8.665	1.34	0.658	0.447	0.332	1.794	0.259	0.106	0.046	0.003
DNS − GARCH − MacroE1	85.69	13.576	7.027	5.190	4.380	17.21	2.703	1.373	0.985	0.798	8.650	1.344	0.666	0.459	0.350	1.803	0.256	0.101	0.039	−0.008
horizon = 12-month ahead																				
DNS	120.24	55.49	21.874	13.88	10.21	23.99	9.131	3.294	2.034	1.480	11.969	3.337	0.972	0.553	0.388	2.344	−1.298	−0.886	−0.632	−0.485
DNS − Macro	120.84	53.57	19.94	12.146	8.716	24.09	8.829	2.989	1.76	1.244	12.00	3.237	0.870	0.462	0.310	2.326	−1.238	−0.825	−0.577	−0.437
DNS − MacroE1	120.72	54.56	20.53	12.469	8.892	24.07	8.985	3.081	1.811	1.272	11.99	3.288	0.901	0.479	0.319	2.329	−1.269	−0.844	−0.587	−0.443
DNS − GARCH	118.88	14.42	3.330	4.670	6.527	23.78	2.648	0.366	0.579	0.898	11.89	1.176	−0.004	0.068	0.195	2.387	−0.001	−0.301	−0.341	−0.368
DNS − GARCH − Macro	119.27	14.72	3.402	4.635	6.398	23.85	2.695	0.377	0.574	0.878	11.92	1.192	−0.001	0.066	0.188	2.375	−0.011	−0.303	−0.340	−0.364
DNS − GARCH − MacroE1	118.99	15.36	4.483	6.039	8.130	23.80	2.796	0.548	0.796	1.151	11.90	1.225	0.056	0.140	0.279	2.384	−0.031	−0.337	−0.384	−0.419

The performance of economic evaluation is evaluated in terms of average utility gain (δ) excess return relative to the risk-free rate [6].

6. Concluding remarks

This research investigate the contribution of forward-looking macroeconomic data, and time-varying yield volatility to the predictive performance of the factor-augmented DNS model. Forward-looking data included are expectations of market participants on inflation, GDP growth, and the level of policy rate. Data on expectations come from the Focus Survey of the Brazilian Central Bank. Results show that the inclusion of these forward-looking variables improves out-of-sample forecast results, specially for shorter maturity bonds.

The inclusion of time-varying volatility also improves out-of-sample forecasts based on statistical measures of predictive ability for certain maturities. However, its value becomes clear when evaluating the economic significance of forecasts. Portfolios formed based on the predictions of models including GARCH effects generate large utility gains for investors regardless of the degree of risk aversion considered.

References

Almeida, Caio Ibsen Rodrigues de, & Faria, Adriano. 2014. Forecasting the Brazilian term structure using macroeconomic factors. *Brazilian Review of Econometrics*, **34**(1), 45–77.

Altavilla, Carlo, Giacomini, Raffaella, & Costantini, Riccardo. 2014. Bond returns and market expectations. *Journal of Financial Econometrics*, **12**(4), 708–729.

Altavilla, Carlo, Giacomini, Raffaella, & Ragusa, Giuseppe. 2017. Anchoring the yield curve using survey expectations. *Journal of Applied Econometrics*, **32**(6), 1055–1068.

Ang, Andrew, & Piazzesi, Monika. 2003. A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Journal of Monetary economics*, **50**(4), 745–787.

Ang, Andrew, Piazzesi, Monika, & Wei, Min. 2006. What does the yield curve tell us about GDP growth? *Journal of econometrics*, **131**(1-2), 359–403.

Bekaert, Geert, Cho, Seonghoon, & Moreno, Antonio. 2010. New Keynesian macroeconomics and the term structure. *Journal of Money, Credit and Banking*, **42**(1), 33–62.

[6] We consider the risk free rate to be the interbank rate CDI.

Bianchi, Francesco, Mumtaz, Haroon, & Surico, Paolo. 2009. The great moderation of the term structure of UK interest rates. *Journal of Monetary Economics*, **56**(6), 856–871.

Bikbov, Ruslan, & Chernov, Mikhail. 2010. No-arbitrage macroeconomic determinants of the yield curve. *Journal of Econometrics*, **159**(1), 166–182.

Bollerslev, Tim. 1986. Generalized autoregressive conditional heteroskedasticity. *Journal of econometrics*, **31**(3), 307–327.

Caldeira, João F, Laurini, Márcio P, & Portugal, Marcelo S. 2010. Bayesian inference applied to dynamic Nelson-Siegel model with stochastic volatility. *Brazilian Review of Econometrics*, **30**(1), 123–161.

Caldeira, João F., Moura, Guilherme V., & Santos, André A. P. 2016. Predicting the Yield Curve Using Forecast Combinations. *Computional Statistics Data Analysis*, **100**(3), 79–98.

Chernov, Mikhail, & Mueller, Philippe. 2012. The term structure of inflation expectations. *Journal of financial economics*, **106**(2), 367–394.

Christensen, Bent Jesper, & van der Wel, Michel. 2019. An asset pricing approach to testing general term structure models. *Journal of Financial Economics*, **134**(1), 165–191.

Chun, Albert Lee. 2011. Expectations, bond yields, and monetary policy. *The Review of Financial Studies*, **24**(1), 208–247.

Chun, Albert Lee. 2012. Forecasting interest rates and inflation: Blue chip clairvoyants or econometrics? *In: EFA 2009 Bergen Meetings Paper.*

Dai, Qiang, & Philippon, Thomas. 2005. *Fiscal policy and the term structure of interest rates.*

de Andrade Alves, Cassio Roberto, Abraham, Kuruvilla Joseph, & Laurini, Marcio Poletti. 2023. Can Brazilian Central Bank Communication help to predict the yield curve? *Journal of Forecasting.*

De Pooter, Michiel, Ravazzolo, Francesco, & Van Dijk, Dick JC. 2010. Term structure forecasting using macro factors and forecast combination. *FRB International Finance Discussion Paper.*

Dewachter, Hans, & Lyrio, Marco. 2006. Macro factors and the term structure of interest rates. *Journal of Money, Credit and Banking*, 119–140.

Dewachter, Hans, Lyrio, Marco, & Maes, Konstantijn. 2006. A joint model for the term structure of interest rates and the macroeconomy. *Journal of Applied Econometrics*, **21**(4), 439–462.

Diebold, Francis X, & Li, Canlin. 2006. Forecasting the term structure of government bond yields. *Journal of econometrics*, **130**(2), 337–364.

Diebold, Francis X, Piazzesi, Monika, & Rudebusch, Glenn D. 2005. Modeling bond yields in finance and macroeconomics. *American Economic Review*, **95**(2), 415–420.

Diebold, Francis X, Rudebusch, Glenn D, & Aruoba, S Boragan. 2006. The macroeconomy and the yield curve: a dynamic latent factor approach. *Journal of econometrics*, **131**(1-2), 309–338.

Duffee, Gregory R. 2006. Term structure estimation without using latent factors. *Journal of Financial Economics*, **79**(3), 507–536.

Duffee, Gregory R. 2011. Information in (and not in) the term structure. *The Review of Financial Studies*, **24**(9), 2895–2934.

Durbin, James, & Koopman, Siem Jan. 2012. *Time series analysis by state space methods*. Vol. 38. Oxford University Press.

Ehling, Paul, Gallmeyer, Michael, Heyerdahl-Larsen, Christian, & Illeditsch, Philipp. 2018. Disagreement about inflation and the yield curve. *Journal of Financial Economics*, **127**(3), 459–484.

Engle, Robert, & Ng, Victor K. 1993. Time-Varying Volatility and the Dynamic Behavior of the Term Structure. *Journal of Money, Credit and Banking*, **25**(3), 336–49.

Exterkate, Peter, Dijk, Dick Van, Heij, Christiaan, & Groenen, Patrick JF. 2013. Forecasting the yield curve in a data-rich environment using the factor-augmented Nelson–Siegel model. *Journal of Forecasting*, **32**(3), 193–214.

Favero, Carlo A, Niu, Linlin, & Sala, Luca. 2012. Term structure forecasting: no-arbitrage restrictions versus large information set. *Journal of Forecasting*, **31**(2), 124–156.

Fernandes, Marcelo, & Vieira, Fausto. 2019. A dynamic Nelson–Siegel model with forward-looking macroeconomic factors for the yield curve in the US. *Journal of Economic Dynamics and Control*, **106**, 103720.

Fleming, Michael J, & Remolona, Eli M. 2001. The term structure of announcement effects.

Gallmeyer, Michael F, Hollifield, Burton, & Zin, Stanley E. 2005. Taylor rules, McCallum rules and the term structure of interest rates. *Journal of Monetary Economics*, **52**(5), 921–950.

Gargano, Antonio, Pettenuzzo, Davide, & Timmermann, Allan. 2019. Bond Return Predictability: Economic Value and Links to the Macroeconomy. *Management Science*, **65**(2), 508–540.

Giacomini, Raffaella, & White, Halbert. 2006. Tests of conditional predictive ability. *Econometrica*, **74**(6), 1545–1578.

Gürkaynak, Refet S, & Wright, Jonathan H. 2012. Macroeconomics and the term structure. *Journal of Economic Literature*, **50**(2), 331–67.

Hansen, Peter R, Lunde, Asger, & Nason, James M. 2011. The model confidence set. *Econometrica*, **79**(2), 453–497.

Harvey, Andrew C, Ruiz, Esther, & Sentana, Enrique. 1992. Unobserved component time series models with ARCH disturbances.

Hautsch, Nikolaus, & Yang, Fuyu. 2012. Bayesian inference in a stochastic volatility Nelson–Siegel model. *Computational Statistics & Data Analysis*, **56**(11), 3774–3792.

Hördahl, Peter, Tristani, Oreste, & Vestin, David. 2006. A joint econometric model of macroeconomic and term-structure dynamics. *Journal of Econometrics*, **131**(1-2), 405–444.

Joslin, Scott, Le, Anh, & Singleton, Kenneth J. 2013. Why Gaussian macro-finance term structure models are (nearly) unconstrained factor-VARs. *Journal of Financial Economics*, **109**(3), 604–622.

Joslin, Scott, Priebsch, Marcel, & Singleton, Kenneth J. 2014. Risk premiums in dynamic term structure models with unspanned macro risks. *The Journal of Finance*, **69**(3), 1197–1233.

Jungbacker, Borus, Koopman, Siem Jan, & Wel, Michel. 2014. Smooth Dynamic Factor Analysis With Application To The Us Term Structure Of Interest Rates. *Journal of Applied Econometrics*, **29**(1), 65–90.

Kim, Don H, & Orphanides, Athanasios. 2012. Term structure estimation with survey data on interest rate forecasts. *Journal of Financial and Quantitative Analysis*, **47**(1), 241–272.

KOOPMAN, SIEM JAN, & VAN DER WEL, MICHEL. 2013. Forecasting the US term structure of interest rates using a macroeconomic smooth dynamic factor model. *International Journal of Forecasting*, **29**(4), 676–694.

KOOPMAN, SIEM JAN, MALLEE, MAX IP, & VAN DER WEL, MICHEL. 2010. Analyzing the term structure of interest rates using the dynamic Nelson–Siegel model with time-varying parameters. *Journal of Business & Economic Statistics*, **28**(3), 329–343.

LAURINI, MÁRCIO POLETTI, & HOTTA, LUIZ KOODI. 2010. Bayesian extensions to diebold-li term structure model. *International Review of Financial Analysis*, **19**(5), 342–350.

LUDVIGSON, SYDNEY C, & NG, SERENA. 2009. Macro factors in bond risk premia. *The Review of Financial Studies*, **22**(12), 5027–5067.

MOENCH, EMANUEL. 2008. Forecasting the yield curve in a data-rich environment: A no-arbitrage factor-augmented VAR approach. *Journal of Econometrics*, **146**(1), 26–43.

NELSON, CHARLES R, & SIEGEL, ANDREW F. 1987. Parsimonious modeling of yield curves. *q Journal of Business*, **60**(4), 473–489.

ORPHANIDES, ATHANASIOS, & WEI, MIN. 2012. Evolving macroeconomic perceptions and the term structure of interest rates. *Journal of Economic Dynamics and Control*, **36**(2), 239–254.

PIAZZESI, MONIKA. 2001. *An econometric model of the yield curve with macroeconomic jump effects.*

PIAZZESI, MONIKA. 2005. Bond yields and the Federal Reserve. *Journal of Political Economy*, **113**(2), 311–344.

PIAZZESI, MONIKA, SCHNEIDER, MARTIN, *et al.* 2009. *Trend and cycle in bond premia.* Vol. 424. Citeseer.

PONCELA, PILAR. 2013. Comments on "Forecasting the US term structure of interest rates using a macroeconomic smooth dynamic factor model" by Koopman and van der Wel. *International Journal of Forecasting*, **29**(4), 695–697.

RAPACH, DAVID, & ZHOU, GUOFU. 2013. Forecasting Stock Returns. *Chap. 0, pages 328–383 of:* ELLIOTT, G., GRANGER, C., & TIMMERMANN, A. (eds), *Handbook of Economic Forecasting.* Handbook of Economic Forecasting, vol. 2. Elsevier.

Rudebusch, Glenn D, & Wu, Tao. 2007. Accounting for a shift in term structure behavior with no-arbitrage and macro-finance models. *Journal of Money, Credit and Banking*, **39**(2-3), 395–422.

Rudebusch, Glenn D, & Wu, Tao. 2008. A macro-finance model of the term structure, monetary policy and the economy. *The Economic Journal*, **118**(530), 906–926.

Sarno, Lucio, Schneider, Paul, & Wagner, Christian. 2016. The economic value of predicting bond risk premia. *Journal of Empirical Finance*, **37**(C), 247–267.

Stark, Tom, *et al.* 2010. Realistic evaluation of real-time forecasts in the Survey of Professional Forecasters. *Federal Reserve Bank of Philadelphia Research Rap, Special Report*, **1**.

Thornton, Daniel L., & Valente, Giorgio. 2012. Out-of-Sample Predictions of Bond Excess Returns and Forward Rates: An Asset Allocation Perspective. *Review of Financial Studies*, **25**(10), 3141–3168.

Van Dijk, Dick, Koopman, Siem Jan, Van der Wel, Michel, & Wright, Jonathan H. 2014. Forecasting interest rates with shifting endpoints. *Journal of Applied Econometrics*, **29**(5), 693–712.

Vieira, Fausto, Fernandes, Marcelo, & Chague, Fernando. 2017. Forecasting the Brazilian yield curve using forward-looking variables. *International Journal of Forecasting*, **33**(1), 121–131.

Welch, Ivo, & Goyal, Amit. 2008. A Comprehensive Look at The Empirical Performance of Equity Premium Prediction. *Review of Financial Studies*, **21**(4), 1455–1508.