Estimation Risk in Conditional Asymmetric Least Squares

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Abstract

This study investigates the estimation risk of conditional Asymmetric Least Squares (ALS) risk measures in the context of heteroskedastic financial time series. Our idea is to reconcile a risk measure that takes into account the time-varying effects of market risk, satisfying the essential mathematical and statistical properties. The conditional expectiles (Newey & Powell 1987) and extremiles (Daouia et al. 2019) are both law-invariant and coherent risk measures conditional to the GARCH-type volatility model framework. To account for the estimation risk, we assess the large sample properties of both conditional estimators. Monte Carlo simulations highlight the effectiveness of the bootstrap approach for estimating conditional extremiles and expectiles, and the robustness of confidence intervals created through the resampling method. Empirical analysis reveals that the one-step-ahead forecast of the conditional extremile outperforms conditional VaR and expectile in terms of exceptions across all selected assets. The confidence bands of the conditional risk measures allow us to adopt an aggressive or conservative risk management strategy, although it may be sensitive to the level of volatility of a given asset.

Keywords: Asymmetric Least Squares, Extremiles, Expectiles, Risk Management, GARCH.

1 Introduction

Risk plays a pivotal role in economics essentially because risk-averse agents smooth their consumption across different states of nature. Literature has shown that agents tend to respond more intensively in face of downside movements than positive outcomes (Kahneman & Tversky 2013). For instance, investors more sensitive to downside losses requires a premium for holding stocks that strongly covary to downside market movements Ang et al. (2006)]. As the nature of the risks changes over time, a reliable framework to assess downside risk is imperative to financial and regulatory decisions, for example, capital allocation and risk management.

A risk measure can be defined as the amount of capital required to make a position with loss acceptable. Value-at-Risk[1] (VaR) and Expected Shortfall (ES) are both leading measures to assess portfolio risk. Despite the VaR popularity, the lack of subadditivity property precludes the diversification of risk (Artzner et al. 1999). Expected Shortfall, on the other hand, enjoys several desirable mathematical properties that made the Basel Committee on Banking Supervision[2] (BCBS) recommend changing the quantitative risk metrics system from VaR to ES in internal market risk models late in 2013. However, Expected Shortfall does not satisfy the elicitability property, meaning that is not possible to compare its performance with different forecast methods (Gneiting 2011).

Furthermore, both VaR and ES are criticized for being tail shape-dependent (Kuan et al. 2009, Daouia et al. (2019)), suggesting that both measures may not be suitable to evaluate downside risk scenarios. In particular, quantile-based ES is considered too conservative,

[1]https://www.msci.com/documents/10199/5915b101-4206-4ba0-aee2-3449d5c7e95a

[2]Fundamental review of the trading book: A revised market risk framework. Basel Committee on Banking Supervision, October 2013

since it is conditional only on the tail event. Conversely, quantile-based VaR is considered too liberal, since investors could no be fully aware of the potential size of the loss of an investment due to stock market crashes. Either way, both measures tend to underestimate or overestimate the exposure of a position to market risks, which may generates inefficiency in the allocation of financial resources.

Rather than relying on conventional probability level interpretation of quantiles subject to excessive optimism or pessimism, we propose to manage the market risk exposure of a portfolio under two Asymmetric Least Squares (ALS) estimators called expectiles (Newey & Powell 1987) and extremiles (Daouia et al. 2019). By definition, both expectiles and extremiles of order $\tau \in (0, 1)$ can be formulated by a minimization problem analog to quantiles, although considering a quadratic loss function.

As a risk measure, Bellini & Di Bernardino (2017) and Daouia et al. (2019) observe that both e_τ and ξ_τ are law-invariant, but also satisfy other appealing mathematical properties such as positive homogeneity, translation invariance, monotonicity, and subadditivity, being coherent risk measures (Artzner et al. 1999).

Therefore, both risk measures have many advantages for quantifying risk, especially because their populational estimator depend on the distance to observations and their probability, which is particularly important for actuarial and portfolio allocation problems (Daouia et al. 2018, Daouia et al. (2019)).

The empirical finance literature has documented compelling evidence that heavy-tailed distributions are associated with autocorrelation between squared returns, suggesting that the volatility of market risk factors changes conditionally on their past values (Bollerslev et al. 1994). Since unconditional approaches potentially neglect the evolution of risk over time, our main contribution is to estimate the time-varying effect of both expectiles and extremiles conditional to heteroskedastic models.

The estimation of risk measures under conditional volatility models is not new to the literature (McNeil & Frey 2000, Christoffersen et al. (2004), Kuester et al. (2006)). Gao & Song (2008) were the first to establish consistency and asymptotic normality for Value-at-Risk (VaR) and Expected Shortfall (ES) conditional to standard GARCH process (Bollerslev 1986). Furthermore, they assess the inaccuracy associated with the estimation process by constructing closed-form confidence intervals to quantify the estimation risk.

Recently, Francq & Zakoïan (2015) introduce the risk parameter estimation for general GARCH-type models such as Nelson (1991), Ding et al. (1993), and Glosten et al. (1993), to name a few. They estimate the conditional volatility model using a generalized Quasi-Maximum Likelihood (QML) based on instrumental densities and derive the asymptotic theory for the conditional VaR[3]. Their approach allows one to derive a confidence interval for all parametric forms of the volatility stable by scaling.

Therefore, we propose to estimate the conditional expectiles and extremiles under the Francq & Zakoïan (2015) framework. To the best of our knowledge, there is no study that quantifies the magnitude of the estimation risk for the conditional Asymmetric Least Squares.

A key challenge in constructing proper confidence intervals for both risk measures emerges from the fact that we do not observe the conditional variance and hence we have to account for its estimation. In the case of extremiles, there is no asymptotic theory for the conditional estimator based on volatility models. The absence of a closed-form expression for the

[3]Throughout the paper, we will refer as conditional VaR, ES, expectile, and extremiles, the risks computed conditional on the past returns.

conditional extremiles asymptotic variance and the lack of studies that investigates the estimation risk of conditional ALS motivate us to assess the large sample properties of both estimators by Monte Carlo simulations.

The Monte Carlo study considers a heteroskedastic data scenario that mimics the most important stylized facts of financial time series. Hence, we simulate eight data-generating process that combines the assumption of thin- and heavy-tailed innovations with two distinct parametrizations of the standard GARCH(1,1) process, considering two sample sizes. To evaluate the precision of the conditional ALS, we estimate the processes through the QML and the bootstrap approach of Pascual et al. (2006).

In all scenarios, we observe that both bootstrap and QMLE empirical distribution of the conditional ALS risk measures are unbiased, regardless of the data-generating process. On the other hand, only the bootstrap technique corrects the excess of kurtosis produced by the QMLE in the first step of the estimation. Therefore, the bootstrap distribution coincides with the simulated distribution, being adequate to provide proper confidence intervals.

In addition, we conduct an empirical analysis to give an assessment about the performance of conditional extremiles and expectiles in implementations. First, we consider an expanding window approach to forecast the ALS risk measures conditional to the standard GARCH(1,1) model using the Gaussian QML. Second, we construct bootstrap confidence bands for each measure using the resampling method of Pascual et al. (2006) and Christoffersen et al. (2004). Third, we backtest the forecasts comparing them with the returns realizations in terms of exceptions.

The backtest results show that the lower (upper) bound of the confidence interval is more liberal (conservative) in terms of violations than its one-step-ahead point estimates for all conditional risk measures. In addition, the conditional extremile outperforms the Value-at-Risk and the expectile, assigning fewer events of exceptions than its competing risk measures for all selected assets. Therefore, the first assessment of the conditional Asymmetric Least Squares indicates that the conditional extremile is a promising candidate to estimate the downside risks in implementations, especially for risk management purposes.

The parper is organized as follows. In Section 2 we discuss the methodology. Section 3 we discuss the Monte Carlo study and the empirical analysis. Section 4 presents the final considerations.

2 Methodology

In this section, I briefly discuss the mathematical properties of leading risk measures. In addition, we follow Newey & Powell (1987) and Daouia et al. (2019) to introduce the expectiles and extremiles risk measures, respectively.

2.1 Risk Measures

Artzner et al. (1999) defines a coherent risk measure as a real-valued random variable $X \in \mathcal{X}$ on a measurable space $(\Omega, \mathcal{F})$, a mapping $\varrho : \mathcal{X} \to \mathbb{R}$ that satisfies the following four axioms:

i. Translation invariance: $\varrho(X + \alpha) = \varrho(X) - \alpha$, for all $\alpha \in \mathbb{R}$ and $X \in \mathcal{X}$;

ii. Positive homogeneity: $\varrho(\alpha X) = \alpha\varrho(X)$, for all $\alpha \geq 0$ and $X \in \mathcal{X}$;

iii. Subadditivity: $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$, for all X, Y, $X + Y \in \mathcal{X}$;

iv. Monotonicity: $\varrho(X) \leq \varrho(Y)$, for all $X, Y \in \mathcal{X}$, with $X \leq Y$.

Acerbi (2002) extend the coherence concept to spectral measures of risk by adding two additional mathematical properties:

v. Law invariance: $\varrho(X) = \varrho(Y)$, if X and Y have the same distribution;

vi. Comonotonic additivity: $\varrho(X + Y) = \varrho(X) + \varrho(Y)$, if X and Y are comonotonic random variables.

Our first measure of interest is the Value-at-Risk (VaR). VaR can be defined as the minimal loss under extraordinary market circumstances:

$$\mathrm{VaR}_\tau(X) \equiv F_X^{-1}(\tau) = \inf\{x \in \mathbb{R} : F_X(x) \geq \tau\} \tag{1}$$

where $\tau \in (0, 1)$ denotes the probability level, X is the loss, and F_X is the Cumulative Distribution. Conversely, a VaR with the confidence level of $(1 - \tau)$ is defined as the possible maximum loss for a given holding period.

VaR is extensively used in the industry, being a subject of interest among academics specialized in actuarial science literature over the last decades, establishing itself as a benchmark in the financial universe. On the other hand, the VaR is not considered a coherent risk measure, since it does not satisfy the subadditivity property (Artzner et al. 1999) and often fails to take into account the size of losses beyond the level τ (Danielsson et al. 2001).

2.2 Conditional Asymmetric Least Squares

It is well-documented in the empirical finance literature that asset returns typically exhibit a mean value approaching zero, a certain degree of skewness, and display a significant excess of kurtosis.

The second moment of the asset returns' distribution, namely return volatility, exhibits positive autocorrelation coefficients, implying that squared returns are dependent across time. As a matter of fact, the magnitude of changes in prices tends to cluster in such a way that shocks on t would influence the volatility for many periods ahead, generating persistency across time.

Other features of the volatility, such as long-memory process (Ding et al. 1993, Baillie et al. (1996)) and leverage effect (Black 1976, Christie (1982)) are also documented in the literature. See Bollerslev et al. (1994) for a good review about stylized facts of series of financial time series.

As defined by Francq & Zakoïan (2015), most conditional volatility models takes the form:

$$\begin{cases} \epsilon_t = \sigma_t \eta_t \\ \sigma_t = \sigma\left(\epsilon_{t-1}, \epsilon_{t-2}, \ldots; \theta_0\right) \end{cases} \tag{2}$$

where (η_t) is a sequence of i.i.d random variables, η_t being independent of $\{\epsilon_u, u < t\}$, $\theta_0 \in \mathbb{R}^m$ is a parameter belonging to a parameter space Θ, and $\sigma : \mathbb{R}^\infty \times \Theta \to (0, \infty)$.

Equation (2) show that the return volatility changes conditionally on their past values. Thus, a reliable specification should take into account the time-varying effects of market risk associated with a given portfolio. Therefore, risk measures must provide an assessment of the future portfolio's worst loss conditional to the most recent information.

Let r denote a risk measure and assume that r satisfy the positively homogeneous (ii) and law-invariant (v) mathematical properties. Then, the risk of ϵ_t conditional on $\{\epsilon_u, u < t\}$ is given by:

$$r_{t-1}(\epsilon_t) = \sigma(\epsilon_{t-1}, \ldots; \theta_0) r(\eta_t) \tag{3}$$

In particular, Newey & Powell (1987) propose a new class of estimator that generalizes the first central moment to all quantile levels called expectiles. The expectile population (e_τ)

can be defined as the minimizer of an asymmetric quadratic loss function in $\tau \in (0,1)$, that is

$$e_\tau = \underset{z\in\mathbb{R}}{\arg\min}\mathbb{E}\left\{|\,\tau - \mathbf{1}_{\{Y\leq\theta\}}|(Y-z)^2\right\} \tag{4}$$

Then, for any $\tau \in (0,1)$, the τ-expectile is given by the unique solution z of

$$\tau\mathbf{E}\left[(Y-z)^+\right] = (1-\tau)\mathbf{E}\left[(Y-z)^-\right] \tag{5}$$

where $x^+ = \max\{x, 0\}$ and $x^- = \min\{x, 0\}$.

Expectiles are the only coherent law-invariant risk measure with elicitability property (Ziegel 2016). In that sense, the expectile becomes a convenient and backtestable measure to assess risk (Bellini & Di Bernardino 2017).

In the GARCH framework, the conditional expectiles can be defined as

$$e_t(\tau) = -\sigma(\epsilon_{t-1}, \ldots; \theta_0)e_\eta(\tau) \tag{6}$$

Daouia et al. (2019) introduce a new asymmetric least squares estimator called extremiles (ξ_τ). The minimization problem of the population extremiles are defined in similar fashion as quantiles, replacing the absolute deviations as well as the check function to a quadratic loss function and a special weight-generating function for the τ-th quantile of Y:

$$\xi_\tau = \underset{\theta\in\mathbb{R}}{\arg\min}\mathbb{E}\left\{J_\tau(F(Y))\cdot\left[|Y-\theta|^2 - |Y|^2\right]\right\} \tag{7}$$

where F is a continuous cumulative distribution function and $J_\tau(\cdot) = K'_\tau(\cdot)$, with

$$K_\tau(t) = \begin{cases} 1-(1-t)^{s(\tau)} & \text{if} \quad 0 < \tau \leqslant 1/2 \\ t^{r(\tau)} & \text{if } 1/2 \leqslant \tau < 1 \end{cases} \tag{8}$$

being a distribution function with support $[0,1]$, and $r(\tau) = s(1-\tau) = \log(1/2)/\log(\tau)$.

Then, for any $\tau \in (0,1)$, the τ-extremile ξ_τ has the following closed-form expression:

$$\xi_\tau = \frac{\mathbb{E}[YJ_\tau(F(Y))]}{\mathbb{E}\left[J_\tau(F(Y)\right]} = \int_0^1 q_t dK_\tau(t) \tag{9}$$

where q_t is the quantile function.

Equation (9) also share standard mathematical properties with expectiles, such law invariance and the coherent risk measure properties of Artzner et al. (1999).

In the GARCH framework, the conditional expectiles can be defined as

$$\xi_t(\tau) = -\sigma(\epsilon_{t-1}, \dots; \theta_0)\xi_\eta(\tau) \tag{10}$$

In general, the quantities in equations (6) and (10) are latent since the parameters in the conditional volatility, the GARCH parameters, and the innovations are unknown. The so-called Filtered Historical Simulation (FHS) rely on the Quasi-Maximum Likelihood (QML) method to consistently estimate the vector of parameters θ and the standardized residuals η_t. In the following, we formally define the QML estimators in the same fashion as Francq & Zakoïan (2015).

Given observations $\epsilon_1, \dots, \epsilon_n$, and arbitrary initial values $\tilde{\epsilon}_i$ for $i \leq 0$, we define

$$\tilde{\sigma}_t(\theta) = \sigma\left(\epsilon_{t-1}, \dots, \epsilon_1, \tilde{\epsilon}_0, \tilde{\epsilon}_{-1}, \dots; \theta\right)$$

which is used to approximate $\sigma_t(\theta) = \sigma\left(\epsilon_{t-1}, \dots, \epsilon_1, \epsilon_0, \epsilon_{-1}, \dots; \theta\right)$. Given an instrumental density $h > 0$, consider the QML criterion

$$\widetilde{Q}_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} g\left(\epsilon_t, \tilde{\sigma}_t(\theta)\right), \quad g(x, \sigma) = \log\left\{\frac{1}{\sigma}h\left(\frac{x}{\sigma}\right)\right\} \tag{11}$$

and let the QMLE

$$\hat{\theta}_n^* = \arg\max_{\theta \in \Theta} \widetilde{Q}_n(\theta) \tag{12}$$

In particular, the estimator in equation (11) is the standard Gaussian QMLE whenever h is the standard Gaussian density ϕ. Let $\hat{\theta}_n^*$ denote the Gaussian QMLE. Then, the parameters of the (2) can be obtained by maximizing the (12).

Given the QML estimates $\hat{\theta}_n^*$, one can estimate the conditional volatility $\hat{\sigma}(\hat{\theta}_n^*)$ and calculate the standardized residuals

$$\hat{\eta}_t = \frac{\epsilon_t}{\hat{\sigma}(\hat{\theta}_n^*)} \tag{13}$$

Then, the ALS risk measures can be estimated using the empirical standardized residuals distribution $\hat{F}_\eta$. Since equation (9) belongs to the class of distortion risk measures (Wang 2000), the τ-extremile of order $\tau \in (0, 1)$ can be estimated by

$$\hat{\xi}_{\hat{\eta}}(\tau) = \int_0^1 \hat{q_\tau} dK_\tau(t) = \sum_{i=1}^{n} \left\{ K_\tau\left(\frac{i}{n}\right) - K_\tau\left(\frac{i-1}{n}\right) \right\} Y_{i,n}$$

where $Y_{1,n} \leq Y_{2,n} \leq \cdots \leq Y_{n,n}$ denotes the ordered sample.

On the other hand, the sample expectiles does not have a closed-form expression. Hence, $\hat{e}_{\hat{\eta}}(\tau)$ is fitted to univariate samples with least asymmetrically weighted squares for asymmetries between 0 and 1 (Sobotka & Kneib 2012).

Naturally, estimation risk arises when computing the measures because we estimate expectiles and extremiles from the observed data. Hence, it is imperative to account the estimation risk, controlling the uncertainty by constructing proper confidence bands.

Berkes & Horváth (2003) demonstrate that GARCH estimation affects the asymptotic behavior of the empirical distribution function and the empirical process of squared residuals. In the case of VaR, Gao & Song (2008) and Francq & Zakoïan (2015) establish that the asymptotic variance of the empirical t-th quantile of the standardized residuals depends of the asymptotic variance of the empirical quantile of the innovations plus a term that depends on the estimation of the GARCH parameters.

Although there is an asymptotic theory developed for conditional expectiles (Girard et al. 2021), the asymptotic behavior of the conditional extremile distribution remains unknown. Hence, we propose to assess the large sample properties of both ALS risk measures by Monte Carlo simulations, relying on the resampling method of Pascual et al. (2006) for heteroskedasticity data, extended to conditional risk measures (Christoffersen et al. 2004).

The bootstrap algorithm for GARCH-based risk measures is given as follows:

1. Estimate the GARCH(1,1) model by QML and compute the standardized residuals $\hat{\eta}_t^* = \epsilon_t^*/\sigma_t^*$, $t = 1, \ldots, T$ where $\widehat{F}_{\hat{\eta}_t}$ is the empirical distribution function of the standard residuals.

2. Generate the bootstrap samples $\epsilon_t^* = \eta_t^* \hat{\sigma}_t^*$, with $\hat{\sigma}_t^2 = \hat{\omega} + \hat{\alpha} L_{t-1}^2 + \hat{\beta}\hat{\sigma}_{t-1}^2$ where η_t^* are random draws with replacement from $\widehat{F}_{\hat{\eta}_t}$ with the initial condition $\hat{\sigma}_1^2 = \hat{\omega}/(1 - \hat{\alpha} - \hat{\beta})$.

3. Compute QMLE for each bootstrap sample: $\hat{\theta}^* = \left(\hat{\omega}^*, \hat{\alpha}^*, \hat{\beta}^*\right)$.

4. Compute the bootstrap estimates of the standard residuals: $\hat{\eta}_t^* = \epsilon_t^*/\sigma_t^*$.

5. Compute the conditional volatility prediction one-step-ahead

$$\hat{\sigma}_{T+1|T}^{*2} = \hat{\omega}^* + \hat{\alpha}^* \epsilon_T^{*2} + \hat{\beta}^* \hat{\sigma}_T^{*2}$$

   given $\epsilon_T^* = \epsilon_T$ and

$$\hat{\sigma}_T^{*2} = \frac{\hat{\omega}^*}{1 - \hat{\alpha}^* - \hat{\beta}^*} + \hat{\alpha}^* \sum_{j=0}^{T-2} \hat{\beta}^{*j} \left(\epsilon_{T-j-1}^2 - \frac{\hat{\omega}^*}{1 - \hat{\alpha}^* - \hat{\beta}^*} \right)$$

6. Compute the bootstrap estimates of the risk measures $\hat{r}_{\eta^*}^*(\tau)$ using the standard residuals $\hat{\eta}_t^*$.

7. Compute the bootstrap estimates of the conditional risk measures $\hat{r}_{T+1}^*(\tau)$.

8. Repeat the steps 1-7 a large number of times ($B = 999$) and obtain a sequence of bootstrap conditional risk measures.

9. Obtain the $100(1-\alpha)\%$ bootstrap confidence interval for the conditional risk measures

$$[Q_{\alpha/2}(r^*_{T+1}(\tau)), Q_{\alpha/2}(r^*_{T+1}(\tau))]$$

As Christoffersen et al. (2004) remark, the bootstrap procedure accounts for the estimation risk in computing $\hat{\sigma}^2_{T+1|T}$ when we replace the estimates $\hat{\theta} = (\hat{\omega}, \hat{\alpha}, \hat{\beta})'$ by their bootstrap estimates $\hat{\theta}^* = (\hat{\omega}^*, \hat{\alpha}^*, \hat{\beta}^*)'$ and compute $\hat{\sigma}^{*2}_{T+1|T}$ instead.

3 Results

3.1 Monte Carlo Simulation

We perform a Monte Carlo study to assess the large sample properties of the asymmetric least squares estimators, considering that the data-generating process (DGP) mimics a data-dependent scenario commonly observed in financial data. For that purpose, we simulate daily returns from a GARCH(1,1) assuming heavy-tailed (t-Student with t_8) and thin-tailed (t-Student with t_{500}) innovation distributions.

Also, we consider that both DGP's sets an unconditional volatility of 20% per year i.e, $\omega = 20^2/252 \times (1 - \alpha - \beta)$ under two distinct scenarios, namely Benchmark and High-persistence, whose GARCH parametrization is given as follows:

1. Benchmark: $\alpha = 0.10$ and $\beta = 0.80$;

2. High-persistence: $\alpha = 0.10$ and $\beta = 0.89$.

For each one of the four DGP considered below

$$\begin{cases} \epsilon_t = \sigma_t \eta_t, & \sigma_t^2 = 0.16 + 0.10\epsilon_{t-1}^2 + 0.80\sigma_{t-1}^2, & \eta_t \sim t_8 \\ \epsilon_t = \sigma_t \eta_t, & \sigma_t^2 = 0.02 + 0.10\epsilon_{t-1}^2 + 0.89\sigma_{t-1}^2, & \eta_t \sim t_8 \\ \epsilon_t = \sigma_t \eta_t, & \sigma_t^2 = 0.16 + 0.10\epsilon_{t-1}^2 + 0.80\sigma_{t-1}^2, & \eta_t \sim t_{500} \\ \epsilon_t = \sigma_t \eta_t, & \sigma_t^2 = 0.02 + 0.10\epsilon_{t-1}^2 + 0.89\sigma_{t-1}^2, & \eta_t \sim t_{500} \end{cases} \tag{14}$$

we draw $S = 10,000$ sample paths of size $T = \{1,500; 2,000\}$, burning the first $1,000$ realizations.

In order to verify the asymptotic behavior of the ALS risk measures, first we evaluate the distribution of both QMLE and bootstrap with respect to the standardized residuals. Then, we investigate the distribution of the conditional risk measures $\xi_t(\tau)$ and $e_t(\tau)$, respectively.

Figure 1 displays the nonparametric density distribution of the τ-extremile of order $\tau = 0.05$ with respect to the four DGP scenarios.

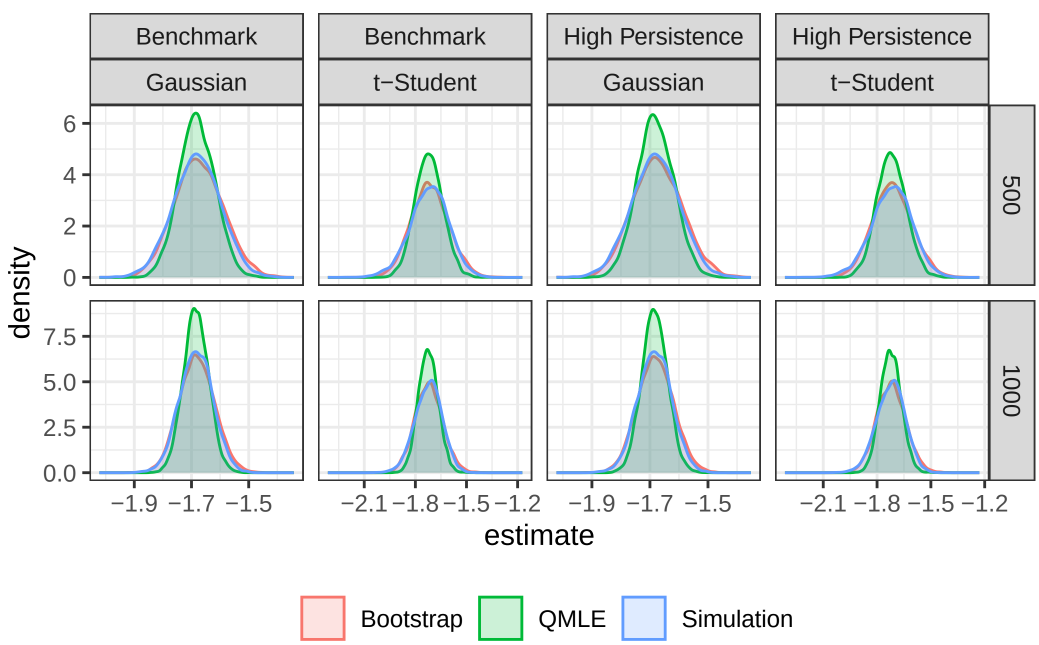

Figure 1: Extremile Distribution of the standardized residuals

Figure 2 shows the nonparametric density distribution of the τ-expectile of order $\tau = 0.05$ with respect to the four DGP scenarios.

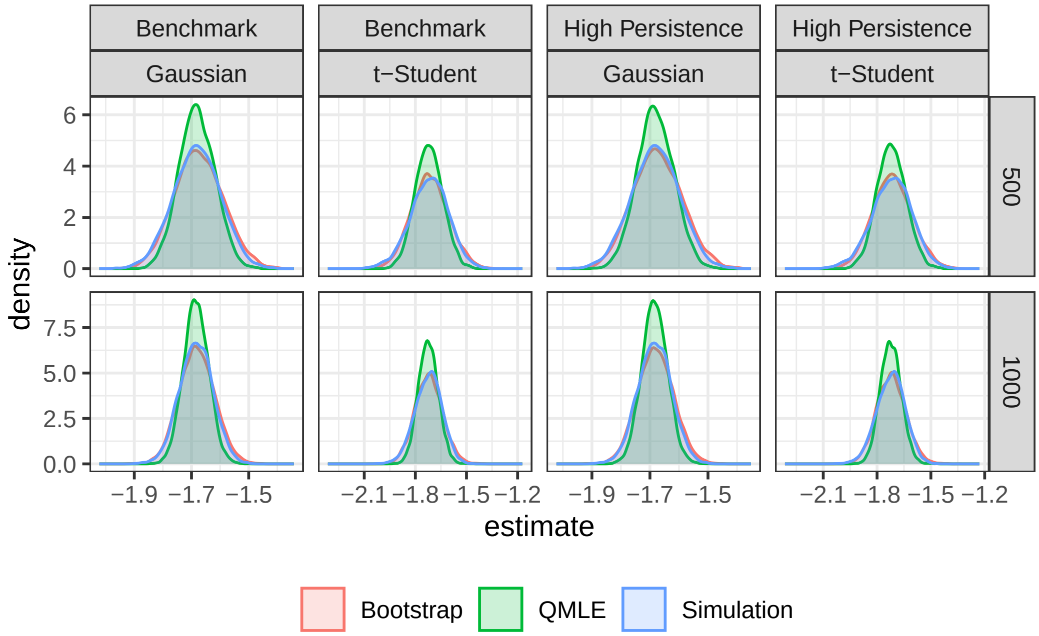

Figure 2: Expectile Distribution of the standardized residuals

In general, we verify in Figure 1 and Figure 2 that a smaller sample size ($T = 500$) adds more dispersion relatively to a larger sample size ($T = 1000$) for all four DGP's. Thus, both bootstrap and QMLE empirical distribution of the conditional ALS risk measures tend to approximate the simulated distribution, regardless of the data-generating process. Therefore, the QMLE and bootstrap estimation of $\xi_\tau(\eta)$ and $e_\tau(\eta)$ seem to be unbiased. On the other hand, only the bootstrap distribution is adequate to provide reliable confidence intervals.

Figure 3 reports the one-step-ahead forecast distribution of the QMLE conditional extremile of order $\tau = 0.05$ considering the four DGP scenarios.

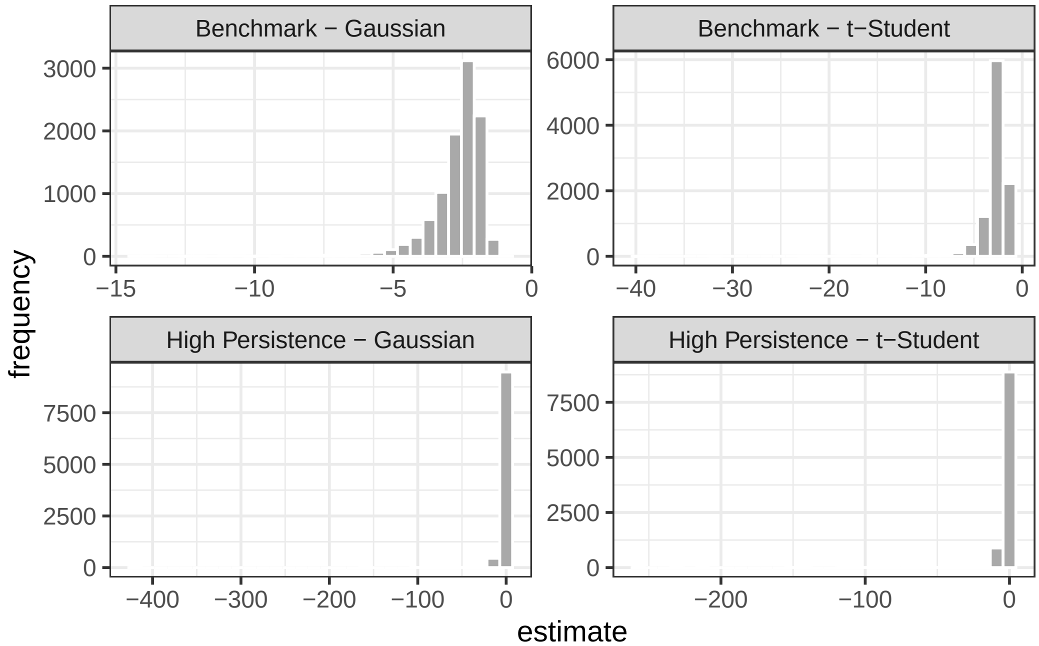

Figure 3: QMLE Conditional Extremile Distribution

Figure 4 shows the one-step-ahead forecast distribution of the QMLE conditional expectile of order $\tau = 0.05$ considering the four DGP scenarios.

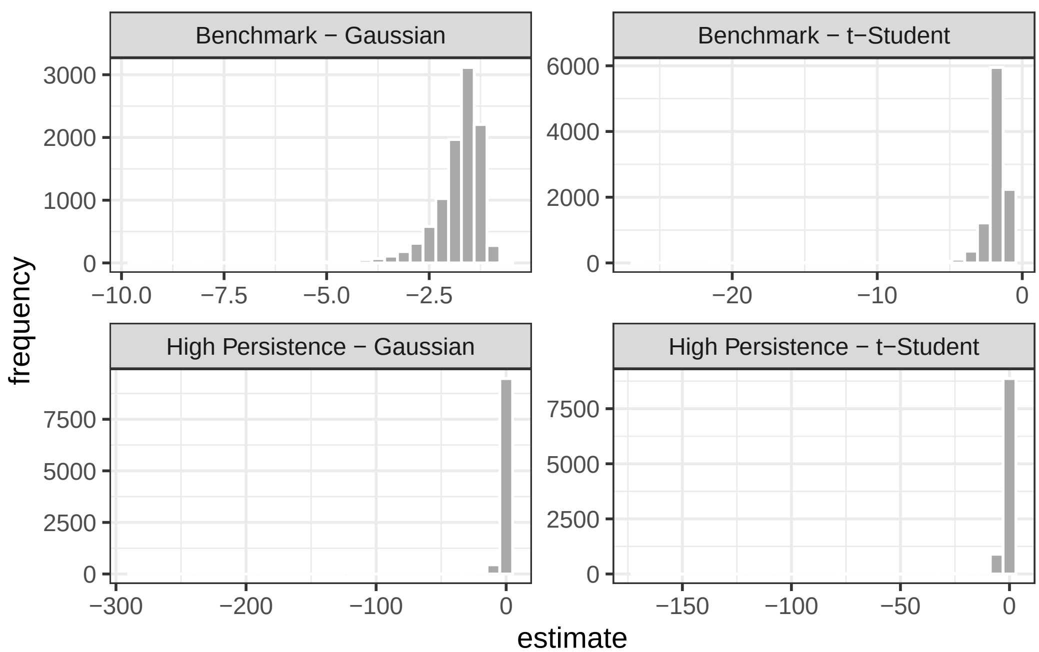

Figure 4: QMLE Conditional Expectile Distribution

As one may observe in Figure 3 and Figure 4, by multiplying the conditional one-step-ahead volatility forecast by a constant amount, say $\xi_\eta(\tau)$ and $e_\eta(\tau)$, would produce a highly skewed empirical distribution for both conditional risk measures. In particular, the higher the persistence of β is, the heavier the tail of the conditional ALS risk measures, whether the innovations are assumed to be Gaussian or t-Student.

3.2 Empirical Results

The empirical analysis is conducted by considering the daily price data of the S&P 500 equity index (SPX), the eurodollar exchange rate (EURUSD), and the most traded cryptocurrencies in terms of market cap such as Bitcoin (BTC), Ethereum (ETH), Binance Coin (BNB), and Dogecoin (DOGE), respectively. The sample period spans approximately 6 years, from September 2017 through June 2023, resulting in 1,510 trading days included in the sample.

Figure 3 shows the evolution of the daily prices of selected assets. Despite the rally of the SPX, the performance of the cryptocurrencies stood out especially after COVID-19, reaching their peak values during the market's surge in late 2021. However, many digital assets experienced significant price corrections in the following year. In particular, Bitcoin reached its all-time high of $\$67,566.83$ on November 8, 2021, although experienced a substantial decline in value, trading at $\$15,787.28$ on November 21, 2022.

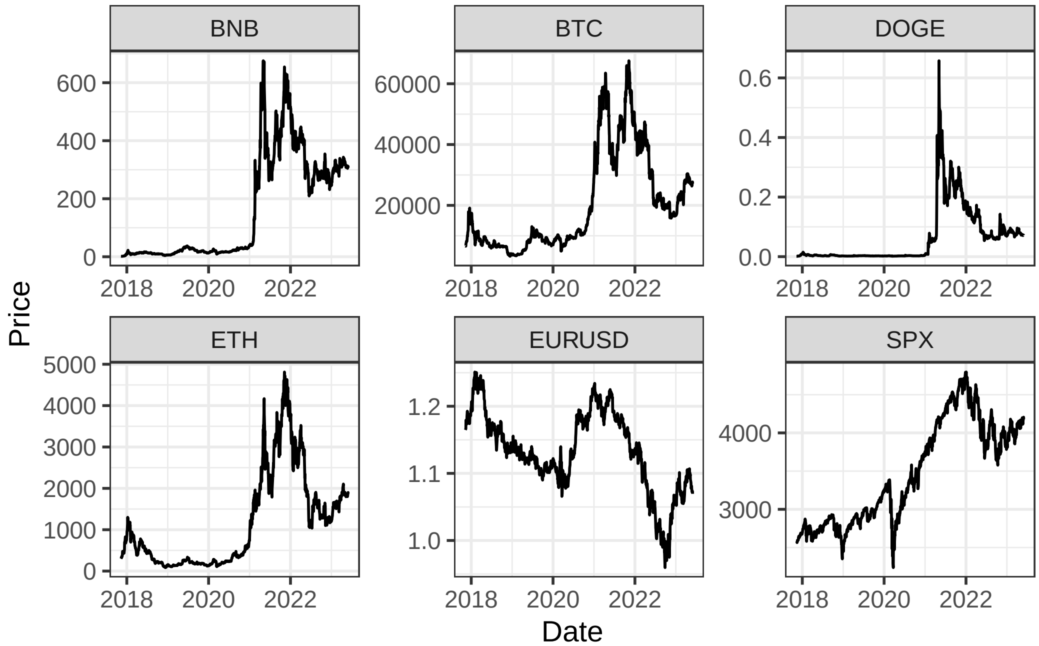

Figure 5: Daily prices time series of selected assets. The historical data of Binance Coin (BNB), Bitcoin (BTC), Dogecoin (DOGE), Ethereum (ETH), eurodollar exchange rate (EURUSD), and S&P 500 equity index (SPX) dates from September 2017 through June 2023.

To evaluate the magnitude of a value drop for each asset, Figure 4 illustrates the boxplot of the historical drawdowns for the S&P 500 and EURUSD compared to selected cryptocurrencies from 2017 to 2023. It is evident from the interquartile range (IQR) standpoint that Ethereum, Dogecoin, Bitcoin, and Binance Coin have experienced significant declines in value with respect to their respective peak values, in contrast to the relatively milder drawdowns of the eurodollar and S&P 500 during this period. These results show the substantial price fluctuations and volatility observed in the cryptocurrency market while highlighting the comparatively more stable performance of traditional financial benchmarks like the S&P 500 index and EURUSD.

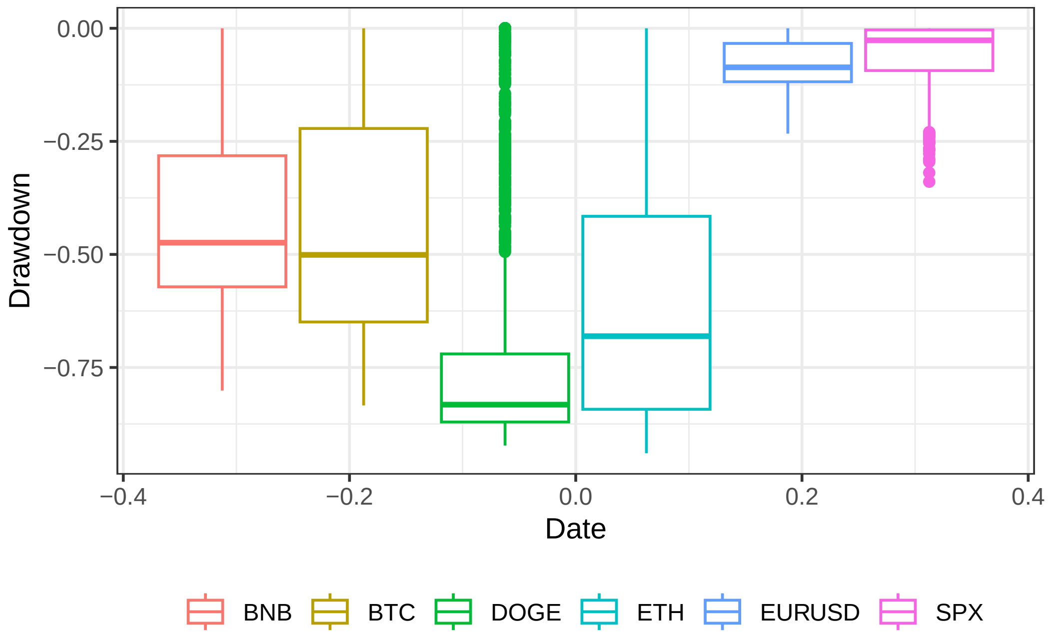

Figure 6: Boxplot of the historical drawdowns from 2017 to 2023. The first quartile (Q1) of Binance Coin (BNB), Bitcoin (BTC), Dogecoin (DOGE), and Ethereum (ETH) is equal to or higher than the upper fence (1.5 times the third quartile (Q3)) of the eurodollar exchange rate (EURUSD), and S&P 500 equity index (SPX).

Table 1 reports the summary statistics of the selected assets. As one should expect, returns are stationary since we rejected (not rejected) the null hypothesis of the Augmented Dickey-Fuller (Kwiatkowski-Phillips-Schmidt-Shin) unit-root (stationary) test for a level of significance $\alpha = 0.05$. Besides, the skewness and kurtosis coefficients suggest that the returns are not normally distributed.

Table 1: Summary statistics of the returns for selected assets

ticker	Min	Q1	Med	Q3	Max	Avg	SD	Skew	Kurt	ADF	KPSS
BNB	-0.42	-0.02	0	0.03	0.70	0.00	0.06	1.96	23.45	-28.79	0.28
BTC	-0.37	-0.02	0	0.02	0.25	0.00	0.04	-0.02	7.03	-32.11	0.22

ticker	Min	Q1	Med	Q3	Max	Avg	SD	Skew	Kurt	ADF	KPSS
DOGE	-0.40	-0.02	0	0.02	3.56	0.01	0.11	18.74	589.57	-23.45	0.16
ETH	-0.42	-0.02	0	0.03	0.26	0.00	0.05	-0.23	5.55	-30.64	0.15
EURUSD	-0.03	0.00	0	0.00	0.02	0.00	0.00	-0.13	1.83	-28.08	0.11
SPX	-0.12	0.00	0	0.01	0.09	0.00	0.01	-0.52	13.35	-11.83	0.06

To evaluate the accuracy of each conditional ALS risk measure, we carry out one-step-ahead sample predictions using a GARCH(1,1) model. This is done by employing an expanding window approach starting with a window size of $W_T = 1,000$, allowing us to refit the model recursively with the Gaussian QML. For backtesting purposes, we reserve the last 510 days of the sample for testing the precision of the conditional extremiles and expectiles, contrasting their results with the conditional VaR of Gao & Song (2008).

Figure 7 contrasts the Historical (dotted line) and Filtered Historical Simulation (dashed line) of the VaR, expectiles, and extremiles estimates from the third quarter of 2022 until the most recent returns realization of the SPX and BTC. As one may observe, there is transitivity among the three risk measures for the selected assets, in which $\hat{e}_t(\tau) < \widehat{\text{VaR}}_t(\tau) < \hat{\xi}_t(\tau)$, meaning that the conditional VaR show itself as a middle course between the expectiles and the extremiles.

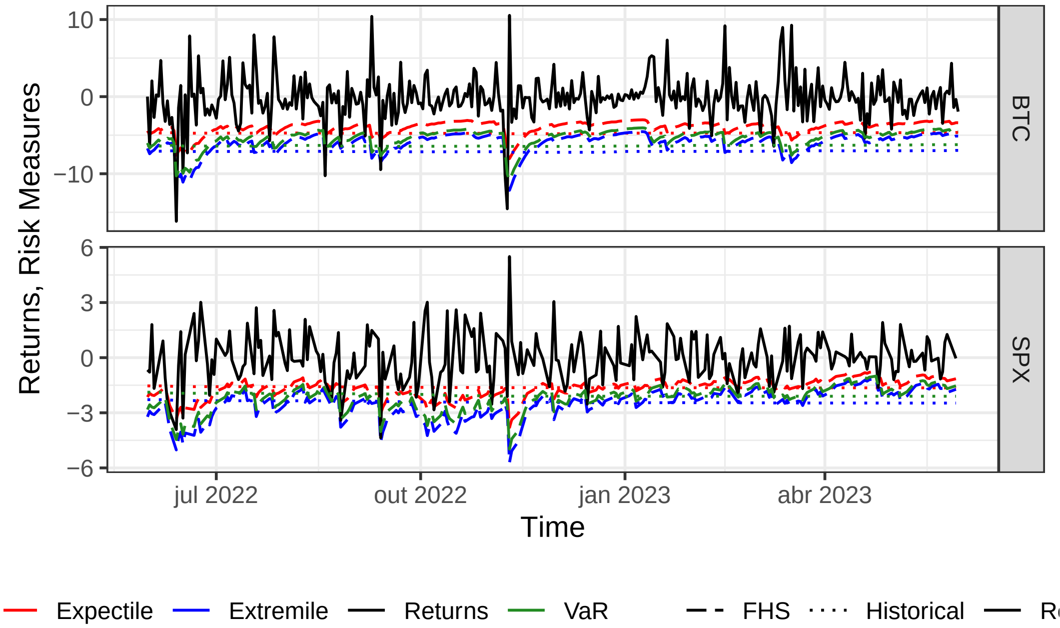

Figure 7: Historical and Filtered Historical Simulation of risk measures for selected assets

In addition, we observe that the historical simulation estimation method completely neglects the time-varying effect of the volatility, in contrast to the Filtered Historical Simulation method. Therefore, the conditional risk measures under the FHS framework present themselves as a reliable alternative to gauge the asset's market risk.

As previously discussed, an alternative to take into account the the parameter uncertainty associated to the FHS estimation process would be the resampling technique of Christoffersen et al. (2004) to construct proper confidence intervals for the extremiles and expectiles. For that purpose, we contrast the daily returns with each of the conditional risk measures with their estimates and confidence bands, but we also provide a backtest procedure based on the number of violations to given an assessement about the performance of the conditional estimators.

Figure 8 shows the bootstrap confidence interval of the conditional extremile of order

$\tau = 0.05$, considering a level of significance of $\alpha = 0.05$ for the S&P 500 equity index and the Bitcoin.

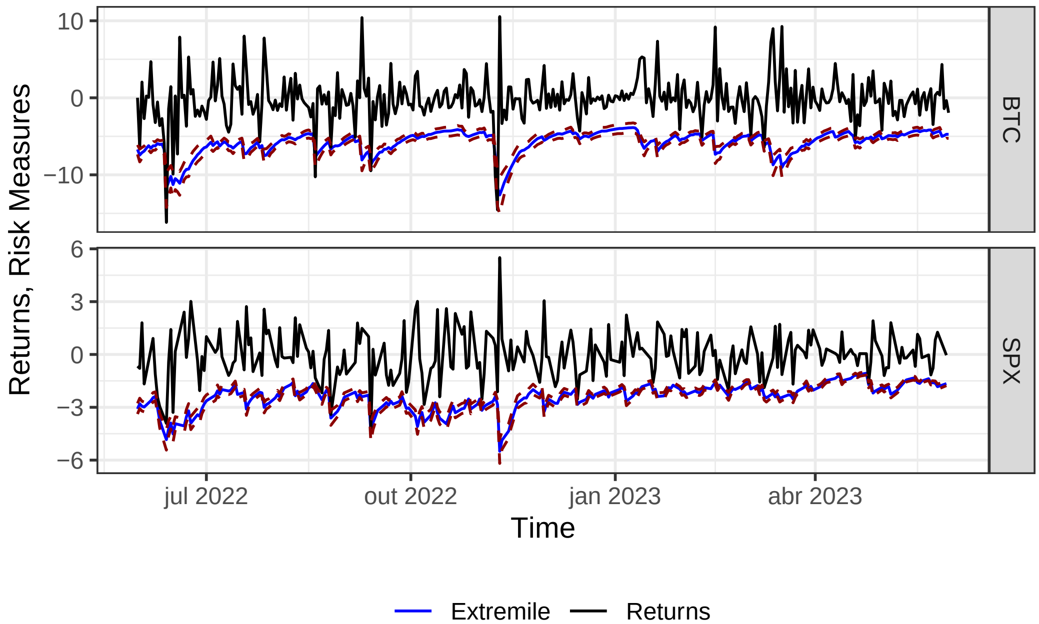

Figure 8: Bootstrap confidence interval of conditional extremiles for selected assets

Likewise, Figure 9 illustrates the bootstrap confidence interval of the conditional expectile of order $\tau = 0.05$, considering a level of significance of $\alpha = 0.05$ for the S&P 500 equity index and the Bitcoin.

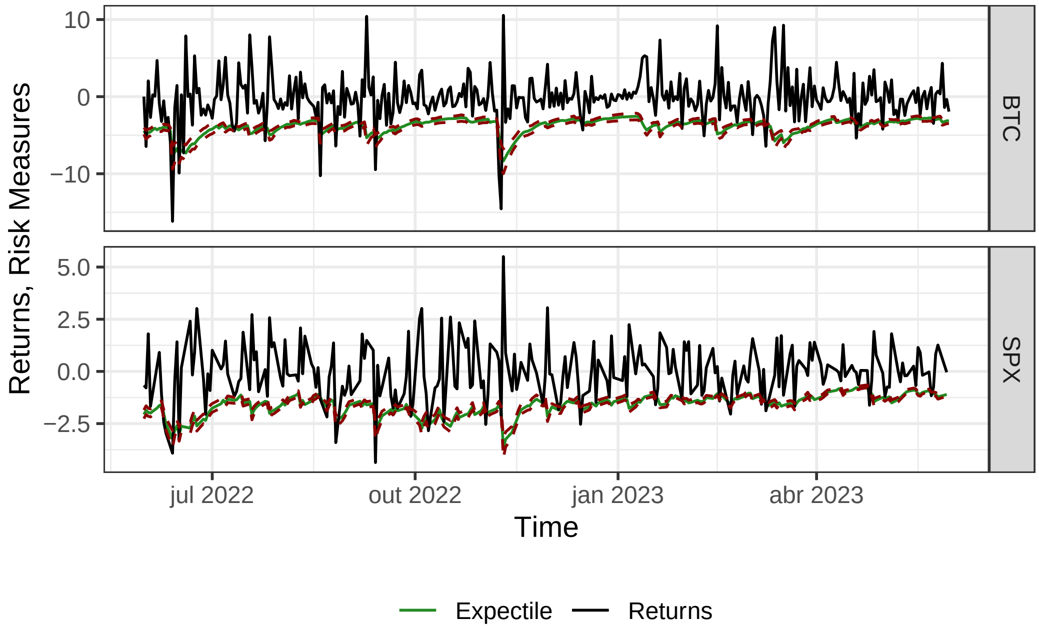

Figure 9: Bootstrap confidence interval of conditional expectiles for selected assets.

Table 2 reports the backtest of the VaR, extremile and expectile at level $\tau = 0.05$ for selected assets, considering a confidence level with level of significance $\alpha = 0.05$. Overall, we verify that the lower bound (upper bound) of the conditional risk measures tends to be much more liberal (conservative) than its point estimates for all selected assets, according to the number of violations. Besides, the confidence bands seem to be sensitive to the level of volatility of a given asset, especially for the DOGE coin, in which we observe several exceptions for the lower bound estimates and none for the upper bound, regardless the conditional risk measure considered.

Table 2: Number of violations of the conditional risk measures one-step-ahead forecasts.

Ticker	Risk Measure	Lower Bound	Estimate	Upper Bound
BNB	Expectile	28	22	17
BNB	Extremile	9	4	2

Ticker	Risk Measure	Lower Bound	Estimate	Upper Bound
BNB	VaR	15	11	4
BTC	Expectile	24	21	19
BTC	Extremile	10	6	4
BTC	VaR	17	15	7
DOGE	Expectile	57	42	0
DOGE	Extremile	27	12	0
DOGE	VaR	33	21	0
ETH	Expectile	29	21	17
ETH	Extremile	13	8	5
ETH	VaR	16	13	8
EURUSD	Expectile	34	30	27
EURUSD	Extremile	14	9	8
EURUSD	VaR	21	14	11
SPX	Expectile	27	22	17
SPX	Extremile	3	0	0
SPX	VaR	7	3	0

Moreover, we observe less exceptions in the conditional extremile estimates than the conditional VaR for all assets considered, in contrast to the expectiles, in which the latter has a poor performance in terms of violations.

Therefore, although both asymmetric least squares risk measures take into account both frequency and the distance to observations in their formulation, the extremile presents itself as a proper candidate to estimate the downside risks in implementations.

4 Final Remark

In this study, we investigate the estimation risk of the conditional Asymmetric Least Squares (ALS) risk measures in the context of heteroskedastic financial time series.

We assess the large sample properties of both conditional extremiles and expectiles by Monte Carlo experiment. Our simulations show that the bootstrap standardized residuals tend to approximate towards the true simulated distribution, regardless of the data-generating process. Moreover, we demonstrated the effectiveness of the resampling method proposed by Christoffersen et al. (2004) in constructing robust confidence intervals for ALS risk measures, especially in the absence of a closed-form expression for the extremiles asymptotic variance.

Also, we conduct an analysis of the performance of conditional extremiles and expectiles in implementations. Empirically, we found that the one-step-ahead predictions of the conditional extremile outperformed both the conditional VaR and expectile in terms of exceptions for all selected assets. Additionally, the confidence bands of the conditional risk measures allow us to adopt an aggressive or conservative risk management strategy, which may be sensitive to the level of volatility of a given asset.

These conditional Asymmetric Least Squares models could also be extended to other univariate GARCH-type models (Francq & Zakoïan 2015). Most important, although the derivation of the limiting distribution for GARCH-based conditional ALS risk measures is not trivial, an alternative to provide closed-form confidence intervals to quantify the estimation risk is to show that the estimator is asymptotically multivariate normal and then apply the delta method. We left this for future research.

References

Acerbi, C. (2002), 'Spectral measures of risk: A coherent representation of subjective risk aversion', *Journal of Banking & Finance* **26**(7), 1505–1518.

Ang, A., Chen, J. & Xing, Y. (2006), 'Downside risk', *The review of financial studies* **19**(4), 1191–1239.

Artzner, P., Delbaen, F., Eber, J.-M. & Heath, D. (1999), 'Coherent measures of risk', *Mathematical Finance* **9**(3), 203–228.

Baillie, R. T., Bollerslev, T. & Mikkelsen, H. O. (1996), 'Fractionally integrated generalized autoregressive conditional heteroskedasticity', *Journal of Econometrics* **74**(1), 3–30.

Bellini, F. & Di Bernardino, E. (2017), 'Risk management with expectiles', *The European Journal of Finance* **23**(6), 487–506.

Berkes, I. & Horváth, L. (2003), 'Limit results for the empirical process of squared residuals in garch models', *Stochastic Processes and their Applications* **105**(2), 271–298.

Black, F. (1976), 'Studies of stock market volatility changes', *1976 Proceedings of the American Statistical Association Business and Economic Statistics Section* .

Bollerslev, T. (1986), 'Generalized autoregressive conditional heteroskedasticity', *Journal of econometrics* **31**(3), 307–327.

Bollerslev, T., Engle, R. F. & Nelson, D. B. (1994), 'Arch models', *Handbook of Econometrics* **4**, 2959–3038.

Christie, A. A. (1982), 'The stochastic behavior of common stock variances: Value, leverage and interest rate effects', *Journal of Financial Economics* **10**(4), 407–432.

Christoffersen, P., Gonçalves, S. et al. (2004), *Estimation risk in financial risk management*, Citeseer.

Danielsson, J., Embrechts, P., Goodhart, C., Keating, C., Muennich, F., Renault, O., Shin, H. S. et al. (2001), 'An academic response to basel ii'.

Daouia, A., Gijbels, I. & Stupfler, G. (2019), 'Extremiles: A new perspective on asymmetric least squares', *Journal of the American Statistical Association* **114**(527), 1366–1381.

Daouia, A., Girard, S. & Stupfler, G. (2018), 'Estimation of tail risk based on extreme expectiles', *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **80**(2), 263–292.

Ding, Z., Granger, C. W. & Engle, R. F. (1993), 'A long memory property of stock market returns and a new model', *Journal of Empirical Finance* **1**(1), 83–106.

Francq, C. & Zakoïan, J.-M. (2015), 'Risk-parameter estimation in volatility models', *Journal of Econometrics* **184**(1), 158–173.

Gao, F. & Song, F. (2008), 'Estimation risk in garch var and es estimates', *Econometric Theory* **24**(5), 1404–1424.

Girard, S., Stupfler, G. & Usseglio-Carleve, A. (2021), 'Extreme conditional expectile estimation in heavy-tailed heteroscedastic regression models', *The Annals of statistics* **49**(6), 3358–3382.

Glosten, L. R., Jagannathan, R. & Runkle, D. E. (1993), 'On the relation between the expected value and the volatility of the nominal excess return on stocks', *The journal of finance* **48**(5), 1779–1801.

Gneiting, T. (2011), 'Making and evaluating point forecasts', *Journal of the American*

Statistical Association **106**(494), 746–762.

URL: *https://doi.org/10.1198/jasa.2011.r10138*

Kahneman, D. & Tversky, A. (2013), Prospect theory: An analysis of decision under risk, *in* 'Handbook of the fundamentals of financial decision making: Part I', World Scientific, pp. 99–127.

Kuan, C.-M., Yeh, J.-H. & Hsu, Y.-C. (2009), 'Assessing value at risk with care, the conditional autoregressive expectile models', *Journal of Econometrics* **150**(2), 261–270.

Kuester, K., Mittnik, S. & Paolella, M. S. (2006), 'Value-at-risk prediction: A comparison of alternative strategies', *Journal of Financial Econometrics* **4**(1), 53–89.

McNeil, A. J. & Frey, R. (2000), 'Estimation of tail-related risk measures for heteroscedastic financial time series: an extreme value approach', *Journal of Empirical Finance* **7**(3-4), 271–300.

Nelson, D. B. (1991), 'Conditional heteroskedasticity in asset returns: A new approach', *Econometrica: Journal of the Econometric Society* pp. 347–370.

Newey, W. K. & Powell, J. L. (1987), 'Asymmetric least squares estimation and testing', *Econometrica* **55**(4), 819–847.

URL: *http://www.jstor.org/stable/1911031*

Pascual, L., Romo, J. & Ruiz, E. (2006), 'Bootstrap prediction for returns and volatilities in garch models', *Computational Statistics & Data Analysis* **50**(9), 2293–2312.

Sobotka, F. & Kneib, T. (2012), 'Geoadditive expectile regression', *Computational Statistics & Data Analysis* **56**(4), 755–767.

Wang, S. S. (2000), 'A class of distortion operators for pricing financial and insurance risks', *Journal of risk and insurance* pp. 15–36.

Ziegel, J. F. (2016), 'Coherence and elicitability', *Mathematical Finance* **26**(4), 901–918.